

Engineering Probability and Statistics

Dispersion, Mean, Median, and Mode Values

If X_1, X_2, \dots, X_n represent the values of a random sample of n items or observations, the *arithmetic mean* of these items or observations, denoted \bar{X} , is defined as

$$\bar{X} = (1/n)(X_1 + X_2 + \dots + X_n) = (1/n) \sum_{i=1}^n X_i$$

$\bar{X} \rightarrow \mu$ for sufficiently large values of n .

The *weighted arithmetic mean* is

$$\bar{X}_w = \frac{\sum w_i X_i}{\sum w_i}$$

where

X_i = the value of the i th observation, and

w_i = the weight applied to X_i .

The *variance* of the population is the *arithmetic mean* of the *squared deviations from the population mean*. If μ is the arithmetic mean of a discrete population of size N , the *population variance* is defined by

$$\begin{aligned} \sigma^2 &= (1/N) \left[(X_1 - \mu)^2 + (X_2 - \mu)^2 + \dots + (X_N - \mu)^2 \right] \\ &= (1/N) \sum_{i=1}^N (X_i - \mu)^2 \end{aligned}$$

Standard deviation formulas (assuming statistical independence) are

$$\sigma_{\text{population}} = \sqrt{(1/N) \sum (X_i - \mu)^2}$$

$$\sigma_{\text{sum}} = \sqrt{\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2}$$

$$\sigma_{\text{series}} = \sigma \sqrt{n}$$

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{n}}$$

$$\sigma_{\text{product}} = \sqrt{A^2 \sigma_b^2 + B^2 \sigma_a^2}$$

The *sample variance* is

$$s^2 = \left[1/(n-1) \right] \sum_{i=1}^n (X_i - \bar{X})^2$$

The *sample standard deviation* is

$$s = \sqrt{\left[1/(n-1) \right] \sum_{i=1}^n (X_i - \bar{X})^2}$$

The *sample coefficient of variation* = $CV = s/\bar{X}$

The *sample geometric mean* = $\sqrt[n]{X_1 X_2 X_3 \dots X_n}$

The *sample root-mean-square value* = $\sqrt{(1/n) \sum X_i^2}$

When the discrete data are rearranged in increasing order and n is odd, the median is the value of the $\left(\frac{n+1}{2}\right)^{\text{th}}$ item

When n is even, the median is the average of the $\left(\frac{n}{2}\right)^{\text{th}}$ and $\left(\frac{n}{2} + 1\right)^{\text{th}}$ items.

The *mode* of a set of data is the value that occurs with greatest frequency.

The *sample range* R is the largest sample value minus the smallest sample value.

Permutations and Combinations

A *permutation* is a particular sequence of a given set of objects. A *combination* is the set itself without reference to order.

1. The number of different *permutations* of n distinct objects *taken r at a time* is

$$P(n, r) = \frac{n!}{(n - r)!}$$

nPr is an alternative notation for $P(n, r)$

2. The number of different *combinations* of n distinct objects *taken r at a time* is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{[r!(n - r)!]}$$

nCr and $\binom{n}{r}$ are alternative notations for $C(n, r)$

3. The number of different *permutations* of n objects *taken n at a time*, given that n_i are of type i , where $i = 1, 2, \dots, k$ and $\sum n_i = n$, is

$$P(n; n_1, n_2, \dots, n_k) = \frac{n!}{n_1! n_2! \dots n_k!}$$

Sets

De Morgan's Law

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Associative Law

$$A \cup (B \cap C) = (A \cup B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup C$$

Distributive Law

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Laws of Probability

Property 1. General Character of Probability

The probability $P(E)$ of an event E is a real number in the range of 0 to 1. The probability of an impossible event is 0 and that of an event certain to occur is 1.

Property 2. Law of Total Probability

$$P(A + B) = P(A) + P(B) - P(A, B)$$

where

$P(A + B)$ = the probability that either A or B occur alone or that both occur together

$P(A)$ = the probability that A occurs

$P(B)$ = the probability that B occurs

$P(A, B)$ = the probability that both A and B occur simultaneously

Property 3. Law of Compound or Joint Probability

If neither $P(A)$ nor $P(B)$ is zero,

$$P(A, B) = P(A)P(B | A) = P(B)P(A | B)$$

where

$P(B | A)$ = the probability that B occurs given the fact that A has occurred

$P(A | B)$ = the probability that A occurs given the fact that B has occurred

If either $P(A)$ or $P(B)$ is zero, then $P(A, B) = 0$.

Bayes' Theorem

$$P(B_j | A) = \frac{P(B_j)P(A | B_j)}{\sum_{i=1}^n P(A | B_i)P(B_i)}$$

where

$P(A_j)$ = the probability of event A_j within the population of A

$P(B_j)$ = the probability of event B_j within the population of B

Probability Functions, Distributions, and Expected Values

A random variable X has a probability associated with each of its possible values. The probability is termed a discrete probability if X can assume only discrete values, or

$$X = x_1, x_2, x_3, \dots, x_n$$

The *discrete probability* of any single event, $X = x_i$, occurring is defined as $P(x_i)$ while the *probability mass function* of the random variable X is defined by

$$f(x_k) = P(X = x_k), k = 1, 2, \dots, n$$

Probability Density Function

If X is continuous, the *probability density function*, f , is defined such that

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

Cumulative Distribution Functions

The *cumulative distribution function*, F , of a discrete random variable X that has a probability distribution described by $P(x_i)$ is defined as

$$F(x_m) = \sum_{k=1}^m P(x_k) = P(X \leq x_m), m = 1, 2, \dots, n$$

If X is continuous, the *cumulative distribution function*, F , is defined by

$$F(x) = \int_{-\infty}^x f(x) dx$$

which implies that $F(a)$ is the probability that $X \leq a$.

Expected Values

Let X be a discrete random variable having a probability mass function

$$f(x_k), k = 1, 2, \dots, n$$

The expected value of X is defined as

$$\mu = E[X] = \sum_{k=1}^n x_k f(x_k)$$

The variance of X is defined as

$$\sigma^2 = V[X] = \sum_{k=1}^n (x_k - \mu)^2 f(x_k)$$

Let X be a continuous random variable having a density function $f(X)$ and let $Y = g(X)$ be some general function. The expected value of Y is:

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) dx$$

The mean or expected value of the random variable X is now defined as

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x) dx$$

while the variance is given by

$$\sigma^2 = V[X] = E[(X - \mu)^2] = E[x^2] - \mu^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

The standard deviation is given by

$$\sigma = \sqrt{V[X]}$$

The coefficient of variation is defined as σ/μ .

Combinations of Random Variables

$$Y = a_1 X_1 + a_2 X_2 + \dots + a_n X_n$$

The expected value of Y is:

$$\mu_y = E(Y) = a_1 E(X_1) + a_2 E(X_2) + \dots + a_n E(X_n)$$

If the random variables are statistically *independent*, then the variance of Y is:

$$\begin{aligned} \sigma_y^2 &= V(Y) = a_1^2 V(X_1) + a_2^2 V(X_2) + \dots + a_n^2 V(X_n) \\ &= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + \dots + a_n^2 \sigma_n^2 \end{aligned}$$

Also, the standard deviation of Y is:

$$\sigma_y = \sqrt{\sigma_y^2}$$

When $Y = f(X_1, X_2, \dots, X_n)$ and X_i are independent, the standard deviation of Y is expressed as:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial X_1} \sigma_{X_1}\right)^2 + \left(\frac{\partial f}{\partial X_2} \sigma_{X_2}\right)^2 + \dots + \left(\frac{\partial f}{\partial X_n} \sigma_{X_n}\right)^2}$$

Binomial Distribution

$P(x)$ is the probability that x successes will occur in n trials.

If p = probability of success and q = probability of failure = $1 - p$, then

$$P_n(x) = C(n, x) p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

where

$$x = 0, 1, 2, \dots, n$$

$C(n, x)$ = number of combinations

n, p = parameters

The variance is given by the form:

$$\sigma^2 = npq$$

Normal Distribution (Gaussian Distribution)

This is a unimodal distribution, the mode being $x = \mu$, with two points of inflection (each located at a distance σ to either side of the mode). The averages of n observations tend to become normally distributed as n increases. The variate x is said to be normally distributed if its density function $f(x)$ is given by an expression of the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where

μ = population mean

σ = standard deviation of the population

$$-\infty \leq x \leq \infty$$

When $\mu = 0$ and $\sigma^2 = \sigma = 1$, the distribution is called a *standardized* or *unit normal* distribution. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ where } -\infty \leq x \leq \infty.$$

A unit normal distribution table is included at the end of this section. In the table, the following notations are utilized:

$F(x)$ = area under the curve from $-\infty$ to x

$R(x)$ = area under the curve from x to ∞

$W(x)$ = area under the curve between $-x$ and x

$$F(-x) = 1 - F(x)$$

It should be noted that for any normal distribution with mean μ and standard deviation σ , the table for the unit normal distribution can be used by utilizing the following transformation:

$$z = \frac{x - \mu}{\sigma}$$

$f(x)$ then becomes $f(z)$, $F(x)$ becomes $F(z)$, etc.

The Central Limit Theorem

Let X_1, X_2, \dots, X_n be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2 . Then for large n , the Central Limit Theorem asserts that the sum

$Y = X_1 + X_2 + \dots + X_n$ is approximately normal.

$$\mu_{\bar{y}} = \mu$$

and the standard deviation

$$\sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$$

t-Distribution

Student's t -distribution has the probability density function given by:

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

where

ν = number of degrees of freedom

n = sample size

$\nu = n - 1$

Γ = gamma function

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$-\infty \leq t \leq \infty$$

A table later in this section gives the values of $t_{\alpha, \nu}$ for values of α and ν . Note that, in view of the symmetry of the t -distribution, $t_{1-\alpha, \nu} = -t_{\alpha, \nu}$

The function for α follows:

$$\alpha = \int_{t_{\alpha, \nu}}^{\infty} f(t) dt$$

χ^2 - Distribution

If Z_1, Z_2, \dots, Z_n are independent unit normal random variables, then

$$\chi^2 = Z_1^2 + Z_2^2 + \dots + Z_n^2$$

is said to have a chi-square distribution with n degrees of freedom.

A table at the end of this section gives values of $\chi_{\alpha, n}^2$ for selected values of α and n .

Gamma Function

$$\Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt, n > 0$$

Propagation of Error

Measurement Error

Measurement error is defined as: *Measured quantity value minus a reference quantity value.* [Source: ISO JCGM 200:2012 definition 2.16]

Sources of errors in measurements arise from imperfections and disturbances in the measurement process, and added noise. One may model a measurement as:

$$x = x_{\text{ref}} + d_{\text{systematic}} + d_{\text{random}}$$

where x is the measurand (value being measured), x_{ref} is the reference value, $d_{\text{systematic}}$ is a disturbance from the measurement process such as a drift or bias, and d_{random} is a disturbance such as random noise.

Linear Combinations

In mathematics, a linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the results (e.g., if z is a linear combination of x and y , then $z = ax+by$ where a and b are constants).

See the section "Combinations of Random Variables" for how variances and standard deviations of random variables combine.

Measurement Uncertainty

Measurement uncertainty is defined as: *A quantitative estimate of the range of values about the reported or measured value in which the true value is believed to lie.* [Source: ISO JCGM 200:2012, definition 2.26]

Given a desired state or measurement y , which is a function of different measured or available states x_i :

$$y = f(x_1, x_2, \dots, x_n)$$

Given the individual states x_i and their standard deviations σ_{x_i} , and assuming that the different x_i are uncorrelated, the Kline-McClintock equation can be used to compute the expected standard uncertainty of y (σ_y) is:

$$\sigma_y = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \sigma_{x_2}^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2 \sigma_{x_n}^2}$$

Expanded uncertainties are typically given at an approximately 95% level of confidence with a coverage factor of $k = 2$. This represents 95% of the area under a Normal probability distribution and is often called 2 sigma.

Linear Regression and Goodness of Fit

Least Squares

$$\hat{y} = \hat{a} + \hat{b}x$$

where

$$\hat{b} = S_{xy}/S_{xx}$$

$$\hat{a} = \bar{y} - \hat{b}\bar{x}$$

$$S_{xy} = \sum_{i=1}^n x_i y_i - (1/n) \left(\sum_{i=1}^n x_i \right) \left(\sum_{i=1}^n y_i \right)$$

$$S_{xx} = \sum_{i=1}^n x_i^2 - (1/n) \left(\sum_{i=1}^n x_i \right)^2$$

$$\bar{y} = (1/n) \left(\sum_{i=1}^n y_i \right)$$

$$\bar{x} = (1/n) \left(\sum_{i=1}^n x_i \right)$$

where

n = sample size

S_{xx} = sum of squares of x

S_{yy} = sum of squares of y

S_{xy} = sum of x - y products

Residual

$$e_i = y_i - \hat{y} = y_i - (\hat{a} + \hat{b}x_i)$$

Standard Error of Estimate (S_e^2):

$$S_e^2 = \frac{S_{xx}S_{yy} - S_{xy}^2}{S_{xx}(n-2)} = MSE$$

where

$$S_{yy} = \sum_{i=1}^n y_i^2 - (1/n) \left(\sum_{i=1}^n y_i \right)^2$$

Confidence Interval for Intercept (\hat{a}):

$$\hat{a} \pm t_{\alpha/2, n-2} \sqrt{\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}} \right) MSE}$$

Confidence Interval for Slope (\hat{b}):

$$\hat{b} \pm t_{\alpha/2, n-2} \sqrt{\frac{MSE}{S_{xx}}}$$

Sample Correlation Coefficient (R) and Coefficient of Determination (R^2):

$$R = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}}$$

$$R^2 = \frac{S_{xy}^2}{S_{xx}S_{yy}}$$

Hypothesis Testing

Let a "dot" subscript indicate summation over the subscript. Thus:

$$y_{i\cdot} = \sum_{j=1}^n y_{ij} \quad \text{and} \quad y_{\cdot\cdot} = \sum_{i=1}^a \sum_{j=1}^n y_{ij}$$

One-Way Analysis of Variance (ANOVA)

Given independent random samples of size n_i from k populations, then:

$$\sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{\cdot\cdot})^2 = \sum_{i=1}^k n_i (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + \sum_{i=1}^k \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i\cdot})^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{error}}$$

If $N =$ total number observations

$$N = \sum_{i=1}^k n_i, \text{ then}$$

$$SS_{\text{total}} = \sum_{i=1}^k \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{N}$$

$$SS_{\text{treatments}} = \sum_{i=1}^k \frac{y_{i\cdot}^2}{n_i} - \frac{y_{\cdot\cdot}^2}{N}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Randomized Complete Block Design

For k treatments and b blocks

$$\sum_{i=1}^k \sum_{j=1}^b (y_{ij} - \bar{y}_{\cdot\cdot})^2 = b \sum_{i=1}^k (\bar{y}_{i\cdot} - \bar{y}_{\cdot\cdot})^2 + k \sum_{j=1}^b (\bar{y}_{\cdot j} - \bar{y}_{\cdot\cdot})^2 + \sum_{i=1}^k \sum_{j=1}^b (\bar{y}_{ij} - \bar{y}_{\cdot j} - \bar{y}_{i\cdot} + \bar{y}_{\cdot\cdot})^2$$

$$SS_{\text{total}} = SS_{\text{treatments}} + SS_{\text{blocks}} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^k \sum_{j=1}^b y_{ij}^2 - \frac{y_{\cdot\cdot}^2}{kb}$$

$$SS_{\text{treatments}} = \frac{1}{b} \sum_{i=1}^k y_{i\cdot}^2 - \frac{y_{\cdot\cdot}^2}{bk}$$

$$SS_{\text{blocks}} = \frac{1}{k} \sum_{j=1}^b y_{\cdot j}^2 - \frac{y_{\cdot\cdot}^2}{bk}$$

$$SS_{\text{error}} = SS_{\text{total}} - SS_{\text{treatments}} - SS_{\text{blocks}}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Two-factor Factorial Designs

For a levels of Factor A, b levels of Factor B, and n repetitions per cell:

$$\sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{...})^2 = bn \sum_{i=1}^a (\bar{y}_{i..} - \bar{y}_{...})^2 + an \sum_{j=1}^b (\bar{y}_{.j.} - \bar{y}_{...})^2 + n \sum_{i=1}^a \sum_{j=1}^b (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n (y_{ijk} - \bar{y}_{ij.})^2$$

$$SS_{\text{total}} = SS_A + SS_B + SS_{AB} + SS_{\text{error}}$$

$$SS_{\text{total}} = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^n y_{ijk}^2 - \frac{y_{...}^2}{abn}$$

$$SS_A = \sum_{i=1}^a \frac{y_{i..}^2}{bn} - \frac{y_{...}^2}{abn}$$

$$SS_B = \sum_{j=1}^b \frac{y_{.j.}^2}{an} - \frac{y_{...}^2}{abn}$$

$$SS_{AB} = \sum_{i=1}^a \sum_{j=1}^b \frac{y_{ij.}^2}{n} - \frac{y_{...}^2}{abn} - SS_A - SS_B$$

$$SS_{\text{error}} = SS_T - SS_A - SS_B - SS_{AB}$$

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

One-Way ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k - 1}$	$\frac{MST}{MSE}$
Error	$N - k$	SS_{error}	$MSE = \frac{SS_{\text{error}}}{N - k}$	
Total	$N - 1$	SS_{total}		

Randomized Complete Block ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Between Treatments	$k - 1$	$SS_{\text{treatments}}$	$MST = \frac{SS_{\text{treatments}}}{k - 1}$	$\frac{MST}{MSE}$
Between Blocks	$n - 1$	SS_{blocks}	$MSB = \frac{SS_{\text{blocks}}}{n - 1}$	$\frac{MSB}{MSE}$
Error	$(k - 1)(n - 1)$	SS_{error}	$MSE = \frac{SS_{\text{error}}}{(k - 1)(n - 1)}$	
Total	$N - 1$	SS_{total}		

Two-Way Factorial ANOVA Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	<i>F</i>
A Treatments	$a - 1$	SS_A	$MSA = \frac{SS_A}{a - 1}$	$\frac{MSA}{MSE}$
B Treatments	$b - 1$	SS_B	$MSB = \frac{SS_B}{b - 1}$	$\frac{MSB}{MSE}$
AB Interaction	$(a - 1)(b - 1)$	SS_{AB}	$MSAB = \frac{SS_{AB}}{(a - 1)(b - 1)}$	$\frac{MSAB}{MSE}$
Error	$ab(n - 1)$	SS_{error}	$MSE = \frac{SS_E}{ab(n - 1)}$	
Total	$abn - 1$	SS_{total}		

Consider an unknown parameter θ of a statistical distribution. Let the null hypothesis be

$$H_0: \mu = \mu_0$$

and let the alternative hypothesis be

$$H_1: \mu \neq \mu_0$$

Rejecting H_0 when it is true is known as a type I error, while accepting H_0 when it is wrong is known as a type II error. Furthermore, the probabilities of type I and type II errors are usually represented by the symbols α and β , respectively:

$$\alpha = \text{probability (type I error)}$$

$$\beta = \text{probability (type II error)}$$

The probability of a type I error is known as the level of significance of the test.

Table A. Tests on Means of Normal Distribution—Variance Known

Hypothesis	Test Statistic	Criteria for Rejection
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$ Z_0 > Z_{\alpha/2}$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	$Z_0 \equiv \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$	$Z_0 < -Z_{\alpha}$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$Z_0 > Z_{\alpha}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 \neq \gamma$		$ Z_0 > Z_{\alpha/2}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 < \gamma$	$Z_0 \equiv \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$	$Z_0 < -Z_{\alpha}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 > \gamma$		$Z_0 > Z_{\alpha}$

Table B. Tests on Means of Normal Distribution—Variance Unknown

Hypothesis	Test Statistic	Criteria for Rejection
$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$		$ t_0 > t_{\alpha/2, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu < \mu_0$	$t_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$	$t_0 < -t_{\alpha, n-1}$
$H_0: \mu = \mu_0$ $H_1: \mu > \mu_0$		$t_0 > t_{\alpha, n-1}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 \neq \gamma$	$t_0 = \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$ Variances equal $v = n_1 + n_2 - 2$	$ t_0 > t_{\alpha/2, v}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 < \gamma$	$t_0 = \frac{\bar{X}_1 - \bar{X}_2 - \gamma}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$ Variances unequal	$t_0 < -t_{\alpha, v}$
$H_0: \mu_1 - \mu_2 = \gamma$ $H_1: \mu_1 - \mu_2 > \gamma$	$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$	$t_0 > t_{\alpha, v}$

In Table B, $s_p^2 = [(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2]/v$

Table C. Tests on Variances of Normal Distribution with Unknown Mean

Hypothesis	Test Statistic	Criteria for Rejection
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 \neq \sigma_0^2$		$\chi_0^2 > \chi_{\alpha/2, n-1}^2$ or $\chi_0^2 < \chi_{1-\alpha/2, n-1}^2$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 < \sigma_0^2$	$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$	$\chi_0^2 < \chi_{1-\alpha, n-1}^2$
$H_0: \sigma^2 = \sigma_0^2$ $H_1: \sigma^2 > \sigma_0^2$		$\chi_0^2 > \chi_{\alpha, n-1}^2$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	$F_0 = \frac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha/2, n_1-1, n_2-1}$ $F_0 < F_{1-\alpha/2, n_1-1, n_2-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 < \sigma_2^2$	$F_0 = \frac{s_2^2}{s_1^2}$	$F_0 > F_{\alpha, n_2-1, n_1-1}$
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	$F_0 = \frac{s_1^2}{s_2^2}$	$F_0 > F_{\alpha, n_1-1, n_2-1}$

Assume that the values of α and β are given. The sample size can be obtained from the following relationships. In (A) and (B), μ_1 is the value assumed to be the true mean.

(A) $H_0: \mu = \mu_0; H_1: \mu \neq \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{\alpha/2}\right) - \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} - Z_{\alpha/2}\right)$$

An approximate result is

$$n \simeq \frac{(Z_{\alpha/2} + Z_{\beta})^2 \sigma^2}{(\mu_1 - \mu_0)^2}$$

(B) $H_0: \mu = \mu_0; H_1: \mu > \mu_0$

$$\beta = \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} + Z_{\alpha}\right)$$

$$n = \frac{(Z_{\alpha} + Z_{\beta})^2 \sigma^2}{(\mu_1 - \mu_0)^2}$$

Confidence Intervals, Sample Distributions and Sample Size

Confidence Interval for the Mean μ of a Normal Distribution

(A) Standard deviation σ is known

$$\bar{X} - Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

(B) Standard deviation σ is not known

$$\bar{X} - t_{\alpha/2} \frac{s}{\sqrt{n}} \leq \mu \leq \bar{X} + t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ corresponds to $n - 1$ degrees of freedom.

Confidence Interval for the Difference Between Two Means μ_1 and μ_2

(A) Standard deviations σ_1 and σ_2 known

$$\bar{X}_1 - \bar{X}_2 - Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + Z_{\alpha/2} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

(B) Standard deviations σ_1 and σ_2 are not known

$$\bar{X}_1 - \bar{X}_2 - t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 \right]}{n_1 + n_2 - 2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + t_{\alpha/2} \sqrt{\frac{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) \left[(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 \right]}{n_1 + n_2 - 2}}$$

where $t_{\alpha/2}$ corresponds to $n_1 + n_2 - 2$ degrees of freedom.

Confidence Intervals for the Variance σ^2 of a Normal Distribution

$$\frac{(n-1)s^2}{\chi_{\alpha/2, n-1}^2} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi_{1-\alpha/2, n-1}^2}$$

Sample Size

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad n = \left[\frac{z_{\alpha/2} \sigma}{\bar{x} - \mu} \right]^2$$

Test Statistics

The following definitions apply.

$$Z_{\text{var}} = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

$$t_{\text{var}} = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

where

Z_{var} = standard normal Z score

t_{var} = sample distribution test statistic

σ = standard deviation

μ_0 = population mean

\bar{X} = hypothesized mean or sample mean

n = sample size

s = computed sample standard deviation

The Z score is applicable when the standard deviation (s) is known. The test statistic is applicable when the standard deviation (s) is computed at time of sampling.

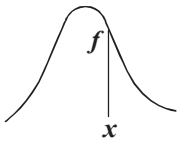
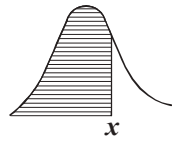
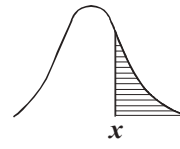
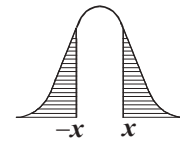
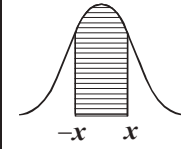
Z_{α} corresponds to the appropriate probability under the normal probability curve for a given Z_{var} .

$t_{\alpha, n-1}$ corresponds to the appropriate probability under the t distribution with $n-1$ degrees of freedom for a given t_{var} .

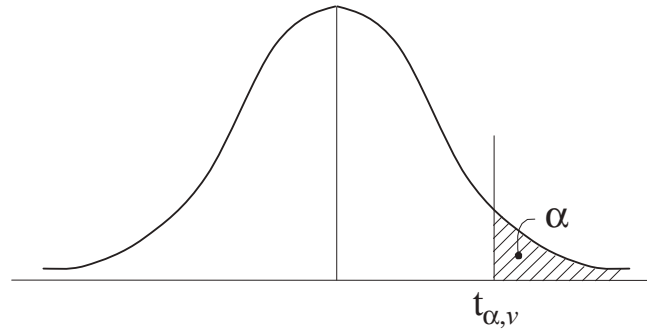
Values of $Z_{\alpha/2}$

Confidence Interval	$Z_{\alpha/2}$
80%	1.2816
90%	1.6449
95%	1.9600
96%	2.0537
98%	2.3263
99%	2.5758

Unit Normal Distribution ($\mu = 0, \sigma = 1$)

					
x	$f(x)$	$F(x)$	$R(x)$	$2R(x)$	$W(x)$
0.0	0.3989	0.5000	0.5000	1.0000	0.0000
0.1	0.3970	0.5398	0.4602	0.9203	0.0797
0.2	0.3910	0.5793	0.4207	0.8415	0.1585
0.3	0.3814	0.6179	0.3821	0.7642	0.2358
0.4	0.3683	0.6554	0.3446	0.6892	0.3108
0.5	0.3521	0.6915	0.3085	0.6171	0.3829
0.6	0.3332	0.7257	0.2743	0.5485	0.4515
0.7	0.3123	0.7580	0.2420	0.4839	0.5161
0.8	0.2897	0.7881	0.2119	0.4237	0.5763
0.9	0.2661	0.8159	0.1841	0.3681	0.6319
1.0	0.2420	0.8413	0.1587	0.3173	0.6827
1.1	0.2179	0.8643	0.1357	0.2713	0.7287
1.2	0.1942	0.8849	0.1151	0.2301	0.7699
1.3	0.1714	0.9032	0.0968	0.1936	0.8064
1.4	0.1497	0.9192	0.0808	0.1615	0.8385
1.5	0.1295	0.9332	0.0668	0.1336	0.8664
1.6	0.1109	0.9452	0.0548	0.1096	0.8904
1.7	0.0940	0.9554	0.0446	0.0891	0.9109
1.8	0.0790	0.9641	0.0359	0.0719	0.9281
1.9	0.0656	0.9713	0.0287	0.0574	0.9426
2.0	0.0540	0.9772	0.0228	0.0455	0.9545
2.1	0.0440	0.9821	0.0179	0.0357	0.9643
2.2	0.0355	0.9861	0.0139	0.0278	0.9722
2.3	0.0283	0.9893	0.0107	0.0214	0.9786
2.4	0.0224	0.9918	0.0082	0.0164	0.9836
2.5	0.0175	0.9938	0.0062	0.0124	0.9876
2.6	0.0136	0.9953	0.0047	0.0093	0.9907
2.7	0.0104	0.9965	0.0035	0.0069	0.9931
2.8	0.0079	0.9974	0.0026	0.0051	0.9949
2.9	0.0060	0.9981	0.0019	0.0037	0.9963
3.0	0.0044	0.9987	0.0013	0.0027	0.9973
Fractiles					
1.2816	0.1755	0.9000	0.1000	0.2000	0.8000
1.6449	0.1031	0.9500	0.0500	0.1000	0.9000
1.9600	0.0584	0.9750	0.0250	0.0500	0.9500
2.0537	0.0484	0.9800	0.0200	0.0400	0.9600
2.3263	0.0267	0.9900	0.0100	0.0200	0.9800
2.5758	0.0145	0.9950	0.0050	0.0100	0.9900

Student's *t*-Distribution

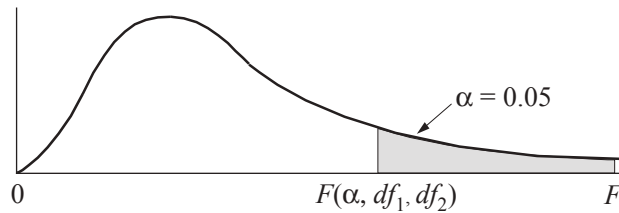


VALUES OF $t_{\alpha, \nu}$

ν	α								ν
	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657	1
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	2
3	0.765	0.978	1.350	1.638	2.353	3.182	4.541	5.841	3
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	4
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	6
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	7
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	8
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	9
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	10
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	11
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	12
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	13
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	14
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	15
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	16
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	17
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	18
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	19
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	20
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	21
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	22
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	23
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	24
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	25
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	26
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	27
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	28
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	29
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	30
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	∞

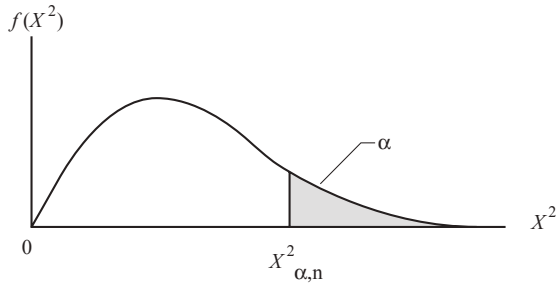
CRITICAL VALUES OF THE F DISTRIBUTION – TABLE

For a particular combination of numerator and denominator degrees of freedom, entry represents the critical values of F corresponding to a specified upper tail area (α).



Denominator df_2	Numerator df_1																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.72	5.69	5.66	5.64	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

CRITICAL VALUES OF X^2 DISTRIBUTION



Degrees of Freedom	$X^2_{.995}$	$X^2_{.990}$	$X^2_{.975}$	$X^2_{.950}$	$X^2_{.900}$	$X^2_{.100}$	$X^2_{.050}$	$X^2_{.025}$	$X^2_{.010}$	$X^2_{.005}$
1	0.0000393	0.0001571	0.0009821	0.0039321	0.0157908	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.0100251	0.0201007	0.0506356	0.102587	0.210720	4.60517	5.99147	7.37776	9.21034	10.5966
3	0.0717212	0.114832	0.215795	0.351846	0.584375	6.25139	7.81473	9.34840	11.3449	12.8381
4	0.206990	0.297110	0.484419	0.710721	1.063623	7.77944	9.48773	11.1433	13.2767	14.8602
5	0.411740	0.554300	0.831211	1.145476	1.61031	9.23635	11.0705	12.8325	15.0863	16.7496
6	0.675727	0.872085	1.237347	1.63539	2.20413	10.6446	12.5916	14.4494	16.8119	18.5476
7	0.989265	1.239043	1.68987	2.16735	2.83311	12.0170	14.0671	16.0128	18.4753	20.2777
8	1.344419	1.646482	2.17973	2.73264	3.48954	13.3616	15.5073	17.5346	20.0902	21.9550
9	1.734926	2.087912	2.70039	3.32511	4.16816	14.6837	16.9190	19.0228	21.6660	23.5893
10	2.15585	2.55821	3.24697	3.94030	4.86518	15.9871	18.3070	20.4831	23.2093	25.1882
11	2.60321	3.05347	3.81575	4.57481	5.57779	17.2750	19.6751	21.9200	24.7250	26.7569
12	3.07382	3.57056	4.40379	5.22603	6.30380	18.5494	21.0261	23.3367	26.2170	28.2995
13	3.56503	4.10691	5.00874	5.89186	7.04150	19.8119	22.3621	24.7356	27.6883	29.8194
14	4.07468	4.66043	5.62872	6.57063	7.78953	21.0642	23.6848	26.1190	29.1413	31.3193
15	4.60094	5.22935	6.26214	7.26094	8.54675	22.3072	24.9958	27.4884	30.5779	32.8013
16	5.14224	5.81221	6.90766	7.96164	9.31223	23.5418	26.2962	28.8454	31.9999	34.2672
17	5.69724	6.40776	7.56418	8.67176	10.0852	24.7690	27.5871	30.1910	33.4087	35.7185
18	6.26481	7.01491	8.23075	9.39046	10.8649	25.9894	28.8693	31.5264	34.8053	37.1564
19	6.84398	7.63273	8.90655	10.1170	11.6509	27.2036	30.1435	32.8523	36.1908	38.5822
20	7.43386	8.26040	9.59083	10.8508	12.4426	28.4120	31.4104	34.1696	37.5662	39.9968
21	8.03366	8.89720	10.28293	11.5913	13.2396	29.6151	32.6705	35.4789	38.9321	41.4010
22	8.64272	9.54249	10.9823	12.3380	14.0415	30.8133	33.9244	36.7807	40.2894	42.7956
23	9.26042	10.19567	11.6885	13.0905	14.8479	32.0069	35.1725	38.0757	41.6384	44.1813
24	9.88623	10.8564	12.4011	13.8484	15.6587	33.1963	36.4151	39.3641	42.9798	45.5585
25	10.5197	11.5240	13.1197	14.6114	16.4734	34.3816	37.6525	40.6465	44.3141	46.9278
26	11.1603	12.1981	13.8439	15.3791	17.2919	35.5631	38.8852	41.9232	45.6417	48.2899
27	11.8076	12.8786	14.5733	16.1513	18.1138	36.7412	40.1133	43.1944	46.9630	49.6449
28	12.4613	13.5648	15.3079	16.9279	18.9392	37.9159	41.3372	44.4607	48.2782	50.9933
29	13.1211	14.2565	16.0471	17.7083	19.7677	39.0875	42.5569	45.7222	49.5879	52.3356
30	13.7867	14.9535	16.7908	18.4926	20.5992	40.2560	43.7729	46.9792	50.8922	53.6720
40	20.7065	22.1643	24.4331	26.5093	29.0505	51.8050	55.7585	59.3417	63.6907	66.7659
50	27.9907	29.7067	32.3574	34.7642	37.6886	63.1671	67.5048	71.4202	76.1539	79.4900
60	35.5346	37.4848	40.4817	43.1879	46.4589	74.3970	79.0819	83.2976	88.3794	91.9517
70	43.2752	45.4418	48.7576	51.7393	55.3290	85.5271	90.5312	95.0231	100.425	104.215
80	51.1720	53.5400	57.1532	60.3915	64.2778	96.5782	101.879	106.629	112.329	116.321
90	59.1963	61.7541	65.6466	69.1260	73.2912	107.565	113.145	118.136	124.116	128.299
100	67.3276	70.0648	74.2219	77.9295	82.3581	118.498	124.342	129.561	135.807	140.169

Source: Thompson, C. M., "Tables of the Percentage Points of the X^2 -Distribution," *Biometrika*, ©1941, 32, 188-189. Reproduced by permission of Oxford University Press.

Cumulative Binomial Probabilities $P(X \leq x)$

		P										
n	x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99
1	0	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000	0.0500	0.0100
2	0	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0025	0.0001
	1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900	0.0975	0.0199
3	0	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010	0.0001	0.0000
	1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280	0.0073	0.0003
	2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710	0.1426	0.0297
4	0	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001	0.0000	0.0000
	1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037	0.0005	0.0000
	2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523	0.0140	0.0006
	3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439	0.1855	0.0394
5	0	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000	0.0000	0.0000
	1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005	0.0000	0.0000
	2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086	0.0012	0.0000
	3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815	0.0226	0.0010
	4	1.0000	0.9997	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095	0.2262	0.0490
6	0	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000	0.0000	0.0000
	1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001	0.0000	0.0000
	2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013	0.0001	0.0000
	3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159	0.0022	0.0000
	4	0.9999	0.9984	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143	0.0328	0.0015
	5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686	0.2649	0.0585
7	0	0.4783	0.2097	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000	0.0000	0.0000
	2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002	0.0000	0.0000
	3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027	0.0002	0.0000
	4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257	0.0038	0.0000
	5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497	0.0444	0.0020
	6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217	0.3017	0.0679
8	0	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000	0.0000	0.0000
	2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000	0.0000	0.0000
	3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004	0.0000	0.0000
	4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050	0.0004	0.0000
	5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381	0.0058	0.0001
	6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869	0.0572	0.0027
	7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695	0.3366	0.0773
9	0	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000	0.0000	0.0000
	2	0.9470	0.7382	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000	0.0000	0.0000
	3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001	0.0000	0.0000
	4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009	0.0000	0.0000
	5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083	0.0006	0.0000
	6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530	0.0084	0.0001
	7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252	0.0712	0.0034
	8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126	0.3698	0.0865

Cumulative Binomial Probabilities $P(X \leq x)$ (continued)

n	x	P											
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95	0.99	
10	0	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000	0.0000	0.0000	0.0000
	3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000	0.0000	0.0000	0.0000
	4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001	0.0000	0.0000	0.0000
	5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016	0.0001	0.0000	0.0000
	6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128	0.0010	0.0000	0.0000
	7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702	0.0115	0.0001	0.0000
	8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639	0.0861	0.0043	0.0000
9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513	0.4013	0.0956	0.0000	
15	0	0.2059	0.0352	0.0047	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5490	0.1671	0.0353	0.0052	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.8159	0.3980	0.1268	0.0271	0.0037	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.9444	0.6482	0.2969	0.0905	0.0176	0.0019	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9873	0.8358	0.5155	0.2173	0.0592	0.0093	0.0007	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9978	0.9389	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	0.0000	0.0000	0.0000	0.0000
	6	0.9997	0.9819	0.8689	0.6098	0.3036	0.0950	0.0152	0.0008	0.0000	0.0000	0.0000	0.0000
	7	1.0000	0.9958	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000	0.0000	0.0000	0.0000
	8	1.0000	0.9992	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003	0.0000	0.0000	0.0000
	9	1.0000	0.9999	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022	0.0001	0.0000	0.0000
	10	1.0000	1.0000	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127	0.0006	0.0000	0.0000
11	1.0000	1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556	0.0055	0.0000	0.0000	
12	1.0000	1.0000	1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841	0.0362	0.0004	0.0000	
13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9948	0.9647	0.8329	0.4510	0.1710	0.0096	0.0000	
14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9953	0.9648	0.7941	0.5367	0.1399	0.0000	
20	0	0.1216	0.0115	0.0008	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.3917	0.0692	0.0076	0.0005	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	2	0.6769	0.2061	0.0355	0.0036	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	3	0.8670	0.4114	0.1071	0.0160	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	4	0.9568	0.6296	0.2375	0.0510	0.0059	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	5	0.9887	0.8042	0.4164	0.1256	0.0207	0.0016	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	6	0.9976	0.9133	0.6080	0.2500	0.0577	0.0065	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	7	0.9996	0.9679	0.7723	0.4159	0.1316	0.0210	0.0013	0.0000	0.0000	0.0000	0.0000	0.0000
	8	0.9999	0.9900	0.8867	0.5956	0.2517	0.0565	0.0051	0.0001	0.0000	0.0000	0.0000	0.0000
	9	1.0000	0.9974	0.9520	0.7553	0.4119	0.1275	0.0171	0.0006	0.0000	0.0000	0.0000	0.0000
	10	1.0000	0.9994	0.9829	0.8725	0.5881	0.2447	0.0480	0.0026	0.0000	0.0000	0.0000	0.0000
	11	1.0000	0.9999	0.9949	0.9435	0.7483	0.4044	0.1133	0.0100	0.0001	0.0000	0.0000	0.0000
	12	1.0000	1.0000	0.9987	0.9790	0.8684	0.5841	0.2277	0.0321	0.0004	0.0000	0.0000	0.0000
	13	1.0000	1.0000	0.9997	0.9935	0.9423	0.7500	0.3920	0.0867	0.0024	0.0000	0.0000	0.0000
	14	1.0000	1.0000	1.0000	0.9984	0.9793	0.8744	0.5836	0.1958	0.0113	0.0003	0.0000	0.0000
15	1.0000	1.0000	1.0000	0.9997	0.9941	0.9490	0.7625	0.3704	0.0432	0.0026	0.0000	0.0000	
16	1.0000	1.0000	1.0000	1.0000	0.9987	0.9840	0.8929	0.5886	0.1330	0.0159	0.0000	0.0000	
17	1.0000	1.0000	1.0000	1.0000	0.9998	0.9964	0.9645	0.7939	0.3231	0.0755	0.0010	0.0000	
18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9924	0.9308	0.6083	0.2642	0.0169	0.0000	
19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9992	0.9885	0.8784	0.6415	0.1821	0.0000	

Montgomery, Douglas C., and George C. Runger, *Applied Statistics and Probability for Engineers*, 4 ed., New York: John Wiley and Sons, 2007.

Statistical Quality Control

Average and Range Charts

n	A_2	D_3	D_4
2	1.880	0	3.268
3	1.023	0	2.574
4	0.729	0	2.282
5	0.577	0	2.114
6	0.483	0	2.004
7	0.419	0.076	1.924
8	0.373	0.136	1.864
9	0.337	0.184	1.816
10	0.308	0.223	1.777

X_i = an individual observation

n = the sample size of a group

k = the number of groups

R = (range) the difference between the largest and smallest observations in a sample of size n .

$$\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_k}{k}$$

$$\bar{R} = \frac{R_1 + R_2 + \dots + R_k}{k}$$

The R Chart formulas are:

$$CL_R = \bar{R}$$

$$UCL_R = D_4 \bar{R}$$

$$LCL_R = D_3 \bar{R}$$

The \bar{X} Chart formulas are:

$$CL_X = \bar{\bar{X}}$$

$$UCL_X = \bar{\bar{X}} + A_2 \bar{R}$$

$$LCL_X = \bar{\bar{X}} - A_2 \bar{R}$$

Standard Deviation Charts

n	A_3	B_3	B_4
2	2.659	0	3.267
3	1.954	0	2.568
4	1.628	0	2.266
5	1.427	0	2.089
6	1.287	0.030	1.970
7	1.182	0.119	1.882
8	1.099	0.185	1.815
9	1.032	0.239	1.761
10	0.975	0.284	1.716

$$UCL_X = \bar{\bar{X}} + A_3 \bar{S}$$

$$CL_X = \bar{\bar{X}}$$

$$LCL_X = \bar{\bar{X}} - A_3 \bar{S}$$

$$UCL_S = B_4 \bar{S}$$

$$CL_S = \bar{S}$$

$$LCL_S = B_3 \bar{S}$$

Approximations

The following table and equations may be used to generate initial approximations of the items indicated.

n	c_4	d_2	d_3
2	0.7979	1.128	0.853
3	0.8862	1.693	0.888
4	0.9213	2.059	0.880
5	0.9400	2.326	0.864
6	0.9515	2.534	0.848
7	0.9594	2.704	0.833
8	0.9650	2.847	0.820
9	0.9693	2.970	0.808
10	0.9727	3.078	0.797

$$\hat{\sigma} = \bar{R} / d_2$$

$$\hat{\sigma} = \bar{S} / c_4$$

$$\sigma_R = d_3 \hat{\sigma}$$

$$\sigma_S = \hat{\sigma} \sqrt{1 - c_4^2}$$

where

$\hat{\sigma}$ = an estimate of σ

σ_R = an estimate of the standard deviation of the ranges of the samples

σ_S = an estimate of the standard deviation of the standard deviations of the samples

Tests for Out of Control

1. A single point falls outside the (three sigma) control limits.
2. Two out of three successive points fall on the same side of and more than two sigma units from the center line.
3. Four out of five successive points fall on the same side of and more than one sigma unit from the center line.
4. Eight successive points fall on the same side of the center line.

Probability and Density Functions: Means and Variances

Variable	Equation	Mean	Variance
Binomial Coefficient	$\binom{n}{x} = \frac{n!}{x!(n-x)!}$		
Binomial	$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Hyper Geometric	$h(x; n, r, N) = \binom{r}{x} \frac{\binom{N-r}{n-x}}{\binom{N}{n}}$	$\frac{nr}{N}$	$\frac{r(N-r)n(N-n)}{N^2(N-1)}$
Poisson	$f(x; \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ
Geometric	$g(x; p) = p(1-p)^{x-1}$	$1/p$	$(1-p)/p^2$
Negative Binomial	$f(y; r, p) = \binom{y+r-1}{r-1} p^r (1-p)^y$	r/p	$r(1-p)/p^2$
Multinomial	$f(x_1, \dots, x_k) = \frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k}$	np_i	$np_i(1-p_i)$
Uniform	$f(x) = 1/(b-a)$	$(a+b)/2$	$(b-a)^2/12$
Gamma	$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}$; $\alpha > 0, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$
Exponential	$f(x) = \frac{1}{\beta} e^{-x/\beta}$	β	β^2
Weibull	$f(x) = \frac{\alpha}{\beta} x^{\alpha-1} e^{-x^\alpha/\beta}$	$\beta^{1/\alpha} \Gamma[(\alpha+1)/\alpha]$	$\beta^{2/\alpha} \left[\Gamma\left(\frac{\alpha+1}{\alpha}\right) - \Gamma^2\left(\frac{\alpha+1}{\alpha}\right) \right]$
Normal	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2
Triangular	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(m-a)} & \text{if } a \leq x \leq m \\ \frac{2(b-x)}{(b-a)(b-m)} & \text{if } m < x \leq b \end{cases}$	$\frac{a+b+m}{3}$	$\frac{a^2 + b^2 + m^2 - ab - am - bm}{18}$

Engineering Economics

Factor Name	Converts	Symbol	Formula
Single Payment Compound Amount	to F given P	$(F/P, i\%, n)$	$(1 + i)^n$
Single Payment Present Worth	to P given F	$(P/F, i\%, n)$	$(1 + i)^{-n}$
Uniform Series Sinking Fund	to A given F	$(A/F, i\%, n)$	$\frac{i}{(1 + i)^n - 1}$
Capital Recovery	to A given P	$(A/P, i\%, n)$	$\frac{i(1 + i)^n}{(1 + i)^n - 1}$
Uniform Series Compound Amount	to F given A	$(F/A, i\%, n)$	$\frac{(1 + i)^n - 1}{i}$
Uniform Series Present Worth	to P given A	$(P/A, i\%, n)$	$\frac{(1 + i)^n - 1}{i(1 + i)^n}$
Uniform Gradient Present Worth	to P given G	$(P/G, i\%, n)$	$\frac{(1 + i)^n - 1}{i^2(1 + i)^n} - \frac{n}{i(1 + i)^n}$
Uniform Gradient † Future Worth	to F given G	$(F/G, i\%, n)$	$\frac{(1 + i)^n - 1}{i^2} - \frac{n}{i}$
Uniform Gradient Uniform Series	to A given G	$(A/G, i\%, n)$	$\frac{1}{i} - \frac{n}{(1 + i)^n - 1}$

Nomenclature and Definitions

A Uniform amount per interest period
 B Benefit
 BV Book value
 C Cost
 d Inflation adjusted interest rate per interest period
 D_j Depreciation in year j
 EV Expected value
 F Future worth, value, or amount
 f General inflation rate per interest period
 G Uniform gradient amount per interest period
 i Interest rate per interest period
 i_e Annual effective interest rate
MARR..... Minimum acceptable/attractive rate of return
 m Number of compounding periods per year
 n Number of compounding periods; or the expected life of an asset
 P Present worth, value, or amount
 r Nominal annual interest rate
 S_n Expected salvage value in year n

Subscripts

j at time j
 n at time n
† $F/G = (F/A - n)/i = (F/A) \times (A/G)$

Non-Annual Compounding

$$i_e = \left(1 + \frac{r}{m}\right)^m - 1$$

Breakeven Analysis

By altering the value of any one of the variables in a situation, holding all of the other values constant, it is possible to find a value for that variable that makes the two alternatives equally economical. This value is the breakeven point.

Breakeven analysis is used to describe the percentage of capacity of operation for a manufacturing plant at which income will just cover expenses.

The payback period is the period of time required for the profit or other benefits of an investment to equal the cost of the investment.

Inflation

To account for inflation, the dollars are deflated by the general inflation rate per interest period f , and then they are shifted over the time scale using the interest rate per interest period i . Use an inflation adjusted interest rate per interest period d for computing present worth values P .

The formula for d is $d = i + f + (i \times f)$

Depreciation

Straight Line

$$D_j = \frac{C - S_n}{n}$$

Modified Accelerated Cost Recovery System (MACRS)

$$D_j = (\text{factor}) C$$

A table of MACRS factors is provided below.

Book Value

$$BV = \text{initial cost} - \sum D_j$$

Taxation

Income taxes are paid at a specific rate on taxable income. Taxable income is total income less depreciation and ordinary expenses. Expenses do not include capital items, which should be depreciated.

Capitalized Costs

Capitalized costs are present worth values using an assumed perpetual period of time.

$$\text{Capitalized Costs} = P = \frac{A}{i}$$

Bonds

Bond value equals the present worth of the payments the purchaser (or holder of the bond) receives during the life of the bond at some interest rate i .

Bond yield equals the computed interest rate of the bond value when compared with the bond cost.

Rate-of-Return

The minimum acceptable rate-of-return (MARR) is that interest rate that one is willing to accept, or the rate one desires to earn on investments. The rate-of-return on an investment is the interest rate that makes the benefits and costs equal.

Benefit-Cost Analysis

In a benefit-cost analysis, the benefits B of a project should exceed the estimated costs C .

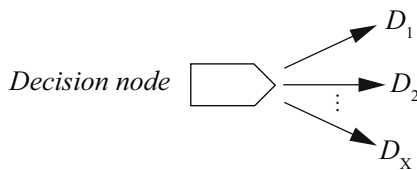
$$B - C \geq 0, \text{ or } B/C \geq 1$$

Modified Accelerated Cost Recovery System (MACRS)

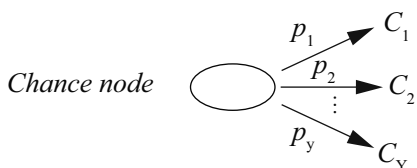
MACRS FACTORS				
Year	Recovery Period (Years)			
	3	5	7	10
	Recovery Rate (Percent)			
1	33.33	20.00	14.29	10.00
2	44.45	32.00	24.49	18.00
3	14.81	19.20	17.49	14.40
4	7.41	11.52	12.49	11.52
5		11.52	8.93	9.22
6		5.76	8.92	7.37
7			8.93	6.55
8			4.46	6.55
9				6.56
10				6.55
11				3.28

Economic Decision Trees

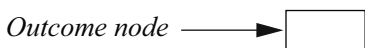
The following symbols are used to model decisions with decision trees:



Decision maker chooses 1 of the available paths.



Represents a probabilistic (chance) event. Each possible outcome (C_1, C_2, \dots, C_Y) has a probability (p_1, p_2, \dots, p_Y) associated with it.



Shows result for a particular path through the decision tree.

Expected Value: $EV = (C_1)(p_1) + (C_2)(p_2) + \dots$

Interest Rate Tables
Factor Table - $i = 0.50\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9950	0.9950	0.0000	1.0050	1.0000	1.0050	1.0000	0.0000
2	0.9901	1.9851	0.9901	1.0100	2.0050	0.5038	0.4988	0.4988
3	0.9851	2.9702	2.9604	1.0151	3.0150	0.3367	0.3317	0.9967
4	0.9802	3.9505	5.9011	1.0202	4.0301	0.2531	0.2481	1.4938
5	0.9754	4.9259	9.8026	1.0253	5.0503	0.2030	0.1980	1.9900
6	0.9705	5.8964	14.6552	1.0304	6.0755	0.1696	0.1646	2.4855
7	0.9657	6.8621	20.4493	1.0355	7.1059	0.1457	0.1407	2.9801
8	0.9609	7.8230	27.1755	1.0407	8.1414	0.1278	0.1228	3.4738
9	0.9561	8.7791	34.8244	1.0459	9.1821	0.1139	0.1089	3.9668
10	0.9513	9.7304	43.3865	1.0511	10.2280	0.1028	0.0978	4.4589
11	0.9466	10.6770	52.8526	1.0564	11.2792	0.0937	0.0887	4.9501
12	0.9419	11.6189	63.2136	1.0617	12.3356	0.0861	0.0811	5.4406
13	0.9372	12.5562	74.4602	1.0670	13.3972	0.0796	0.0746	5.9302
14	0.9326	13.4887	86.5835	1.0723	14.4642	0.0741	0.0691	6.4190
15	0.9279	14.4166	99.5743	1.0777	15.5365	0.0694	0.0644	6.9069
16	0.9233	15.3399	113.4238	1.0831	16.6142	0.0652	0.0602	7.3940
17	0.9187	16.2586	128.1231	1.0885	17.6973	0.0615	0.0565	7.8803
18	0.9141	17.1728	143.6634	1.0939	18.7858	0.0582	0.0532	8.3658
19	0.9096	18.0824	160.0360	1.0994	19.8797	0.0553	0.0503	8.8504
20	0.9051	18.9874	177.2322	1.1049	20.9791	0.0527	0.0477	9.3342
21	0.9006	19.8880	195.2434	1.1104	22.0840	0.0503	0.0453	9.8172
22	0.8961	20.7841	214.0611	1.1160	23.1944	0.0481	0.0431	10.2993
23	0.8916	21.6757	233.6768	1.1216	24.3104	0.0461	0.0411	10.7806
24	0.8872	22.5629	254.0820	1.1272	25.4320	0.0443	0.0393	11.2611
25	0.8828	23.4456	275.2686	1.1328	26.5591	0.0427	0.0377	11.7407
30	0.8610	27.7941	392.6324	1.1614	32.2800	0.0360	0.0310	14.1265
40	0.8191	36.1722	681.3347	1.2208	44.1588	0.0276	0.0226	18.8359
50	0.7793	44.1428	1,035.6966	1.2832	56.6452	0.0227	0.0177	23.4624
60	0.7414	51.7256	1,448.6458	1.3489	69.7700	0.0193	0.0143	28.0064
100	0.6073	78.5426	3,562.7934	1.6467	129.3337	0.0127	0.0077	45.3613

Factor Table - $i = 1.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9901	0.9901	0.0000	1.0100	1.0000	1.0100	1.0000	0.0000
2	0.9803	1.9704	0.9803	1.0201	2.0100	0.5075	0.4975	0.4975
3	0.9706	2.9410	2.9215	1.0303	3.0301	0.3400	0.3300	0.9934
4	0.9610	3.9020	5.8044	1.0406	4.0604	0.2563	0.2463	1.4876
5	0.9515	4.8534	9.6103	1.0510	5.1010	0.2060	0.1960	1.9801
6	0.9420	5.7955	14.3205	1.0615	6.1520	0.1725	0.1625	2.4710
7	0.9327	6.7282	19.9168	1.0721	7.2135	0.1486	0.1386	2.9602
8	0.9235	7.6517	26.3812	1.0829	8.2857	0.1307	0.1207	3.4478
9	0.9143	8.5650	33.6959	1.0937	9.3685	0.1167	0.1067	3.9337
10	0.9053	9.4713	41.8435	1.1046	10.4622	0.1056	0.0956	4.4179
11	0.8963	10.3676	50.8067	1.1157	11.5668	0.0965	0.0865	4.9005
12	0.8874	11.2551	60.5687	1.1268	12.6825	0.0888	0.0788	5.3815
13	0.8787	12.1337	71.1126	1.1381	13.8093	0.0824	0.0724	5.8607
14	0.8700	13.0037	82.4221	1.1495	14.9474	0.0769	0.0669	6.3384
15	0.8613	13.8651	94.4810	1.1610	16.0969	0.0721	0.0621	6.8143
16	0.8528	14.7179	107.2734	1.1726	17.2579	0.0679	0.0579	7.2886
17	0.8444	15.5623	120.7834	1.1843	18.4304	0.0643	0.0543	7.7613
18	0.8360	16.3983	134.9957	1.1961	19.6147	0.0610	0.0510	8.2323
19	0.8277	17.2260	149.8950	1.2081	20.8109	0.0581	0.0481	8.7017
20	0.8195	18.0456	165.4664	1.2202	22.0190	0.0554	0.0454	9.1694
21	0.8114	18.8570	181.6950	1.2324	23.2392	0.0530	0.0430	9.6354
22	0.8034	19.6604	198.5663	1.2447	24.4716	0.0509	0.0409	10.0998
23	0.7954	20.4558	216.0660	1.2572	25.7163	0.0489	0.0389	10.5626
24	0.7876	21.2434	234.1800	1.2697	26.9735	0.0471	0.0371	11.0237
25	0.7798	22.0232	252.8945	1.2824	28.2432	0.0454	0.0354	11.4831
30	0.7419	25.8077	355.0021	1.3478	34.7849	0.0387	0.0277	13.7557
40	0.6717	32.8347	596.8561	1.4889	48.8864	0.0305	0.0205	18.1776
50	0.6080	39.1961	879.4176	1.6446	64.4632	0.0255	0.0155	22.4363
60	0.5504	44.9550	1,192.8061	1.8167	81.6697	0.0222	0.0122	26.5333
100	0.3697	63.0289	2,605.7758	2.7048	170.4814	0.0159	0.0059	41.3426

Interest Rate Tables
Factor Table - $i = 1.50\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9852	0.9852	0.0000	1.0150	1.0000	1.0150	1.0000	0.0000
2	0.9707	1.9559	0.9707	1.0302	2.0150	0.5113	0.4963	0.4963
3	0.9563	2.9122	2.8833	1.0457	3.0452	0.3434	0.3284	0.9901
4	0.9422	3.8544	5.7098	1.0614	4.0909	0.2594	0.2444	1.4814
5	0.9283	4.7826	9.4229	1.0773	5.1523	0.2091	0.1941	1.9702
6	0.9145	5.6972	13.9956	1.0934	6.2296	0.1755	0.1605	2.4566
7	0.9010	6.5982	19.4018	1.1098	7.3230	0.1516	0.1366	2.9405
8	0.8877	7.4859	26.6157	1.1265	8.4328	0.1336	0.1186	3.4219
9	0.8746	8.3605	32.6125	1.1434	9.5593	0.1196	0.1046	3.9008
10	0.8617	9.2222	40.3675	1.1605	10.7027	0.1084	0.0934	4.3772
11	0.8489	10.0711	48.8568	1.1779	11.8633	0.0993	0.0843	4.8512
12	0.8364	10.9075	58.0571	1.1956	13.0412	0.0917	0.0767	5.3227
13	0.8240	11.7315	67.9454	1.2136	14.2368	0.0852	0.0702	5.7917
14	0.8118	12.5434	78.4994	1.2318	15.4504	0.0797	0.0647	6.2582
15	0.7999	13.3432	89.6974	1.2502	16.6821	0.0749	0.0599	6.7223
16	0.7880	14.1313	101.5178	1.2690	17.9324	0.0708	0.0558	7.1839
17	0.7764	14.9076	113.9400	1.2880	19.2014	0.0671	0.0521	7.6431
18	0.7649	15.6726	126.9435	1.3073	20.4894	0.0638	0.0488	8.0997
19	0.7536	16.4262	140.5084	1.3270	21.7967	0.0609	0.0459	8.5539
20	0.7425	17.1686	154.6154	1.3469	23.1237	0.0582	0.0432	9.0057
21	0.7315	17.9001	169.2453	1.3671	24.4705	0.0559	0.0409	9.4550
22	0.7207	18.6208	184.3798	1.3876	25.8376	0.0537	0.0387	9.9018
23	0.7100	19.3309	200.0006	1.4084	27.2251	0.0517	0.0367	10.3462
24	0.6995	20.0304	216.0901	1.4295	28.6335	0.0499	0.0349	10.7881
25	0.6892	20.7196	232.6310	1.4509	30.0630	0.0483	0.0333	11.2276
30	0.6398	24.0158	321.5310	1.5631	37.5387	0.0416	0.0266	13.3883
40	0.5513	29.9158	524.3568	1.8140	54.2679	0.0334	0.0184	17.5277
50	0.4750	34.9997	749.9636	2.1052	73.6828	0.0286	0.0136	21.4277
60	0.4093	39.3803	988.1674	2.4432	96.2147	0.0254	0.0104	25.0930
100	0.2256	51.6247	1,937.4506	4.4320	228.8030	0.0194	0.0044	37.5295

Factor Table - $i = 2.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9804	0.9804	0.0000	1.0200	1.0000	1.0200	1.0000	0.0000
2	0.9612	1.9416	0.9612	1.0404	2.0200	0.5150	0.4950	0.4950
3	0.9423	2.8839	2.8458	1.0612	3.0604	0.3468	0.3268	0.9868
4	0.9238	3.8077	5.6173	1.0824	4.1216	0.2626	0.2426	1.4752
5	0.9057	4.7135	9.2403	1.1041	5.2040	0.2122	0.1922	1.9604
6	0.8880	5.6014	13.6801	1.1262	6.3081	0.1785	0.1585	2.4423
7	0.8706	6.4720	18.9035	1.1487	7.4343	0.1545	0.1345	2.9208
8	0.8535	7.3255	24.8779	1.1717	8.5830	0.1365	0.1165	3.3961
9	0.8368	8.1622	31.5720	1.1951	9.7546	0.1225	0.1025	3.8681
10	0.8203	8.9826	38.9551	1.2190	10.9497	0.1113	0.0913	4.3367
11	0.8043	9.7868	46.9977	1.2434	12.1687	0.1022	0.0822	4.8021
12	0.7885	10.5753	55.6712	1.2682	13.4121	0.0946	0.0746	5.2642
13	0.7730	11.3484	64.9475	1.2936	14.6803	0.0881	0.0681	5.7231
14	0.7579	12.1062	74.7999	1.3195	15.9739	0.0826	0.0626	6.1786
15	0.7430	12.8493	85.2021	1.3459	17.2934	0.0778	0.0578	6.6309
16	0.7284	13.5777	96.1288	1.3728	18.6393	0.0737	0.0537	7.0799
17	0.7142	14.2919	107.5554	1.4002	20.0121	0.0700	0.0500	7.5256
18	0.7002	14.9920	119.4581	1.4282	21.4123	0.0667	0.0467	7.9681
19	0.6864	15.6785	131.8139	1.4568	22.8406	0.0638	0.0438	8.4073
20	0.6730	16.3514	144.6003	1.4859	24.2974	0.0612	0.0412	8.8433
21	0.6598	17.0112	157.7959	1.5157	25.7833	0.0588	0.0388	9.2760
22	0.6468	17.6580	171.3795	1.5460	27.2990	0.0566	0.0366	9.7055
23	0.6342	18.2922	185.3309	1.5769	28.8450	0.0547	0.0347	10.1317
24	0.6217	18.9139	199.6305	1.6084	30.4219	0.0529	0.0329	10.5547
25	0.6095	19.5235	214.2592	1.6406	32.0303	0.0512	0.0312	10.9745
30	0.5521	22.3965	291.7164	1.8114	40.5681	0.0446	0.0246	13.0251
40	0.4529	27.3555	461.9931	2.2080	60.4020	0.0366	0.0166	16.8885
50	0.3715	31.4236	642.3606	2.6916	84.5794	0.0318	0.0118	20.4420
60	0.3048	34.7609	823.6975	3.2810	114.0515	0.0288	0.0088	23.6961
100	0.1380	43.0984	1,464.7527	7.2446	312.2323	0.0232	0.0032	33.9863

Interest Rate Tables
Factor Table - $i = 4.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9615	0.9615	0.0000	1.0400	1.0000	1.0400	1.0000	0.0000
2	0.9246	1.8861	0.9246	1.0816	2.0400	0.5302	0.4902	0.4902
3	0.8890	2.7751	2.7025	1.1249	3.1216	0.3603	0.3203	0.9739
4	0.8548	3.6299	5.2670	1.1699	4.2465	0.2755	0.2355	1.4510
5	0.8219	4.4518	8.5547	1.2167	5.4163	0.2246	0.1846	1.9216
6	0.7903	5.2421	12.5062	1.2653	6.6330	0.1908	0.1508	2.3857
7	0.7599	6.0021	17.0657	1.3159	7.8983	0.1666	0.1266	2.8433
8	0.7307	6.7327	22.1806	1.3686	9.2142	0.1485	0.1085	3.2944
9	0.7026	7.4353	27.8013	1.4233	10.5828	0.1345	0.0945	3.7391
10	0.6756	8.1109	33.8814	1.4802	12.0061	0.1233	0.0833	4.1773
11	0.6496	8.7605	40.3772	1.5395	13.4864	0.1141	0.0741	4.6090
12	0.6246	9.3851	47.2477	1.6010	15.0258	0.1066	0.0666	5.0343
13	0.6006	9.9856	54.4546	1.6651	16.6268	0.1001	0.0601	5.4533
14	0.5775	10.5631	61.9618	1.7317	18.2919	0.0947	0.0547	5.8659
15	0.5553	11.1184	69.7355	1.8009	20.0236	0.0899	0.0499	6.2721
16	0.5339	11.6523	77.7441	1.8730	21.8245	0.0858	0.0458	6.6720
17	0.5134	12.1657	85.9581	1.9479	23.6975	0.0822	0.0422	7.0656
18	0.4936	12.6593	94.3498	2.0258	25.6454	0.0790	0.0390	7.4530
19	0.4746	13.1339	102.8933	2.1068	27.6712	0.0761	0.0361	7.8342
20	0.4564	13.5903	111.5647	2.1911	29.7781	0.0736	0.0336	8.2091
21	0.4388	14.0292	120.3414	2.2788	31.9692	0.0713	0.0313	8.5779
22	0.4220	14.4511	129.2024	2.3699	34.2480	0.0692	0.0292	8.9407
23	0.4057	14.8568	138.1284	2.4647	36.6179	0.0673	0.0273	9.2973
24	0.3901	15.2470	147.1012	2.5633	39.0826	0.0656	0.0256	9.6479
25	0.3751	15.6221	156.1040	2.6658	41.6459	0.0640	0.0240	9.9925
30	0.3083	17.2920	201.0618	3.2434	56.0849	0.0578	0.0178	11.6274
40	0.2083	19.7928	286.5303	4.8010	95.0255	0.0505	0.0105	14.4765
50	0.1407	21.4822	361.1638	7.1067	152.6671	0.0466	0.0066	16.8122
60	0.0951	22.6235	422.9966	10.5196	237.9907	0.0442	0.0042	18.6972
100	0.0198	24.5050	563.1249	50.5049	1,237.6237	0.0408	0.0008	22.9800

Factor Table - $i = 6.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9434	0.9434	0.0000	1.0600	1.0000	1.0600	1.0000	0.0000
2	0.8900	1.8334	0.8900	1.1236	2.0600	0.5454	0.4854	0.4854
3	0.8396	2.6730	2.5692	1.1910	3.1836	0.3741	0.3141	0.9612
4	0.7921	3.4651	4.9455	1.2625	4.3746	0.2886	0.2286	1.4272
5	0.7473	4.2124	7.9345	1.3382	5.6371	0.2374	0.1774	1.8836
6	0.7050	4.9173	11.4594	1.4185	6.9753	0.2034	0.1434	2.3304
7	0.6651	5.5824	15.4497	1.5036	8.3938	0.1791	0.1191	2.7676
8	0.6274	6.2098	19.8416	1.5938	9.8975	0.1610	0.1010	3.1952
9	0.5919	6.8017	24.5768	1.6895	11.4913	0.1470	0.0870	3.6133
10	0.5584	7.3601	29.6023	1.7908	13.1808	0.1359	0.0759	4.0220
11	0.5268	7.8869	34.8702	1.8983	14.9716	0.1268	0.0668	4.4213
12	0.4970	8.3838	40.3369	2.0122	16.8699	0.1193	0.0593	4.8113
13	0.4688	8.8527	45.9629	2.1329	18.8821	0.1130	0.0530	5.1920
14	0.4423	9.2950	51.7128	2.2609	21.0151	0.1076	0.0476	5.5635
15	0.4173	9.7122	57.5546	2.3966	23.2760	0.1030	0.0430	5.9260
16	0.3936	10.1059	63.4592	2.5404	25.6725	0.0990	0.0390	6.2794
17	0.3714	10.4773	69.4011	2.6928	28.2129	0.0954	0.0354	6.6240
18	0.3505	10.8276	75.3569	2.8543	30.9057	0.0924	0.0324	6.9597
19	0.3305	11.1581	81.3062	3.0256	33.7600	0.0896	0.0296	7.2867
20	0.3118	11.4699	87.2304	3.2071	36.7856	0.0872	0.0272	7.6051
21	0.2942	11.7641	93.1136	3.3996	39.9927	0.0850	0.0250	7.9151
22	0.2775	12.0416	98.9412	3.6035	43.3923	0.0830	0.0230	8.2166
23	0.2618	12.3034	104.7007	3.8197	46.9958	0.0813	0.0213	8.5099
24	0.2470	12.5504	110.3812	4.0489	50.8156	0.0797	0.0197	8.7951
25	0.2330	12.7834	115.9732	4.2919	54.8645	0.0782	0.0182	9.0722
30	0.1741	13.7648	142.3588	5.7435	79.0582	0.0726	0.0126	10.3422
40	0.0972	15.0463	185.9568	10.2857	154.7620	0.0665	0.0065	12.3590
50	0.0543	15.7619	217.4574	18.4202	290.3359	0.0634	0.0034	13.7964
60	0.0303	16.1614	239.0428	32.9877	533.1282	0.0619	0.0019	14.7909
100	0.0029	16.6175	272.0471	339.3021	5,638.3681	0.0602	0.0002	16.3711

Interest Rate Tables
Factor Table - $i = 8.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9259	0.9259	0.0000	1.0800	1.0000	1.0800	1.0000	0.0000
2	0.8573	1.7833	0.8573	1.1664	2.0800	0.5608	0.4808	0.4808
3	0.7938	2.5771	2.4450	1.2597	3.2464	0.3880	0.3080	0.9487
4	0.7350	3.3121	4.6501	1.3605	4.5061	0.3019	0.2219	1.4040
5	0.6806	3.9927	7.3724	1.4693	5.8666	0.2505	0.1705	1.8465
6	0.6302	4.6229	10.5233	1.5869	7.3359	0.2163	0.1363	2.2763
7	0.5835	5.2064	14.0242	1.7138	8.9228	0.1921	0.1121	2.6937
8	0.5403	5.7466	17.8061	1.8509	10.6366	0.1740	0.0940	3.0985
9	0.5002	6.2469	21.8081	1.9990	12.4876	0.1601	0.0801	3.4910
10	0.4632	6.7101	25.9768	2.1589	14.4866	0.1490	0.0690	3.8713
11	0.4289	7.1390	30.2657	2.3316	16.6455	0.1401	0.0616	4.2395
12	0.3971	7.5361	34.6339	2.5182	18.9771	0.1327	0.0527	4.5957
13	0.3677	7.9038	39.0463	2.7196	21.4953	0.1265	0.0465	4.9402
14	0.3405	8.2442	43.4723	2.9372	24.2149	0.1213	0.0413	5.2731
15	0.3152	8.5595	47.8857	3.1722	27.1521	0.1168	0.0368	5.5945
16	0.2919	8.8514	52.2640	3.4259	30.3243	0.1130	0.0330	5.9046
17	0.2703	9.1216	56.5883	3.7000	33.7502	0.1096	0.0296	6.2037
18	0.2502	9.3719	60.8426	3.9960	37.4502	0.1067	0.0267	6.4920
19	0.2317	9.6036	65.0134	4.3157	41.4463	0.1041	0.0241	6.7697
20	0.2145	9.8181	69.0898	4.6610	45.7620	0.1019	0.0219	7.0369
21	0.1987	10.0168	73.0629	5.0338	50.4229	0.0998	0.0198	7.2940
22	0.1839	10.2007	76.9257	5.4365	55.4568	0.0980	0.0180	7.5412
23	0.1703	10.3711	80.6726	5.8715	60.8933	0.0964	0.0164	7.7786
24	0.1577	10.5288	84.2997	6.3412	66.7648	0.0950	0.0150	8.0066
25	0.1460	10.6748	87.8041	6.8485	73.1059	0.0937	0.0137	8.2254
30	0.0994	11.2578	103.4558	10.0627	113.2832	0.0888	0.0088	9.1897
40	0.0460	11.9246	126.0422	21.7245	259.0565	0.0839	0.0039	10.5699
50	0.0213	12.2335	139.5928	46.9016	573.7702	0.0817	0.0017	11.4107
60	0.0099	12.3766	147.3000	101.2571	1,253.2133	0.0808	0.0008	11.9015
100	0.0005	12.4943	155.6107	2,199.7613	27,484.5157	0.0800		12.4545

Factor Table - $i = 10.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.9091	0.9091	0.0000	1.1000	1.0000	1.1000	1.0000	0.0000
2	0.8264	1.7355	0.8264	1.2100	2.1000	0.5762	0.4762	0.4762
3	0.7513	2.4869	2.3291	1.3310	3.3100	0.4021	0.3021	0.9366
4	0.6830	3.1699	4.3781	1.4641	4.6410	0.3155	0.2155	1.3812
5	0.6209	3.7908	6.8618	1.6105	6.1051	0.2638	0.1638	1.8101
6	0.5645	4.3553	9.6842	1.7716	7.7156	0.2296	0.1296	2.2236
7	0.5132	4.8684	12.7631	1.9487	9.4872	0.2054	0.1054	2.6216
8	0.4665	5.3349	16.0287	2.1436	11.4359	0.1874	0.0874	3.0045
9	0.4241	5.7590	19.4215	2.3579	13.5735	0.1736	0.0736	3.3724
10	0.3855	6.1446	22.8913	2.5937	15.9374	0.1627	0.0627	3.7255
11	0.3505	6.4951	26.3962	2.8531	18.5312	0.1540	0.0540	4.0641
12	0.3186	6.8137	29.9012	3.1384	21.3843	0.1468	0.0468	4.3884
13	0.2897	7.1034	33.3772	3.4523	24.5227	0.1408	0.0408	4.6988
14	0.2633	7.3667	36.8005	3.7975	27.9750	0.1357	0.0357	4.9955
15	0.2394	7.6061	40.1520	4.1772	31.7725	0.1315	0.0315	5.2789
16	0.2176	7.8237	43.4164	4.5950	35.9497	0.1278	0.0278	5.5493
17	0.1978	8.0216	46.5819	5.0545	40.5447	0.1247	0.0247	5.8071
18	0.1799	8.2014	49.6395	5.5599	45.5992	0.1219	0.0219	6.0526
19	0.1635	8.3649	52.5827	6.1159	51.1591	0.1195	0.0195	6.2861
20	0.1486	8.5136	55.4069	6.7275	57.2750	0.1175	0.0175	6.5081
21	0.1351	8.6487	58.1095	7.4002	64.0025	0.1156	0.0156	6.7189
22	0.1228	8.7715	60.6893	8.1403	71.4027	0.1140	0.0140	6.9189
23	0.1117	8.8832	63.1462	8.9543	79.5430	0.1126	0.0126	7.1085
24	0.1015	8.9847	65.4813	9.8497	88.4973	0.1113	0.0113	7.2881
25	0.0923	9.0770	67.6964	10.8347	98.3471	0.1102	0.0102	7.4580
30	0.0573	9.4269	77.0766	17.4494	164.4940	0.1061	0.0061	8.1762
40	0.0221	9.7791	88.9525	45.2593	442.5926	0.1023	0.0023	9.0962
50	0.0085	9.9148	94.8889	117.3909	1,163.9085	0.1009	0.0009	9.5704
60	0.0033	9.9672	97.7010	304.4816	3,034.8164	0.1003	0.0003	9.8023
100	0.0001	9.9993	99.9202	13,780.6123	137,796.1234	0.1000		9.9927

Interest Rate Tables
Factor Table - $i = 12.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.8929	0.8929	0.0000	1.1200	1.0000	1.1200	1.0000	0.0000
2	0.7972	1.6901	0.7972	1.2544	2.1200	0.5917	0.4717	0.4717
3	0.7118	2.4018	2.2208	1.4049	3.3744	0.4163	0.2963	0.9246
4	0.6355	3.0373	4.1273	1.5735	4.7793	0.3292	0.2092	1.3589
5	0.5674	3.6048	6.3970	1.7623	6.3528	0.2774	0.1574	1.7746
6	0.5066	4.1114	8.9302	1.9738	8.1152	0.2432	0.1232	2.1720
7	0.4523	4.5638	11.6443	2.2107	10.0890	0.2191	0.0991	2.5515
8	0.4039	4.9676	14.4714	2.4760	12.2997	0.2013	0.0813	2.9131
9	0.3606	5.3282	17.3563	2.7731	14.7757	0.1877	0.0677	3.2574
10	0.3220	5.6502	20.2541	3.1058	17.5487	0.1770	0.0570	3.5847
11	0.2875	5.9377	23.1288	3.4785	20.6546	0.1684	0.0484	3.8953
12	0.2567	6.1944	25.9523	3.8960	24.1331	0.1614	0.0414	4.1897
13	0.2292	6.4235	28.7024	4.3635	28.0291	0.1557	0.0357	4.4683
14	0.2046	6.6282	31.3624	4.8871	32.3926	0.1509	0.0309	4.7317
15	0.1827	6.8109	33.9202	5.4736	37.2797	0.1468	0.0268	4.9803
16	0.1631	6.9740	36.3670	6.1304	42.7533	0.1434	0.0234	5.2147
17	0.1456	7.1196	38.6973	6.8660	48.8837	0.1405	0.0205	5.4353
18	0.1300	7.2497	40.9080	7.6900	55.7497	0.1379	0.0179	5.6427
19	0.1161	7.3658	42.9979	8.6128	63.4397	0.1358	0.0158	5.8375
20	0.1037	7.4694	44.9676	9.6463	72.0524	0.1339	0.0139	6.0202
21	0.0926	7.5620	46.8188	10.8038	81.6987	0.1322	0.0122	6.1913
22	0.0826	7.6446	48.5543	12.1003	92.5026	0.1308	0.0108	6.3514
23	0.0738	7.7184	50.1776	13.5523	104.6029	0.1296	0.0096	6.5010
24	0.0659	7.7843	51.6929	15.1786	118.1552	0.1285	0.0085	6.6406
25	0.0588	7.8431	53.1046	17.0001	133.3339	0.1275	0.0075	6.7708
30	0.0334	8.0552	58.7821	29.9599	241.3327	0.1241	0.0041	7.2974
40	0.0107	8.2438	65.1159	93.0510	767.0914	0.1213	0.0013	7.8988
50	0.0035	8.3045	67.7624	289.0022	2,400.0182	0.1204	0.0004	8.1597
60	0.0011	8.3240	68.8100	897.5969	7,471.6411	0.1201	0.0001	8.2664
100		8.3332	69.4336	83,522.2657	696,010.5477	0.1200		8.3321

Factor Table - $i = 18.00\%$

n	P/F	P/A	P/G	F/P	F/A	A/P	A/F	A/G
1	0.8475	0.8475	0.0000	1.1800	1.0000	1.1800	1.0000	0.0000
2	0.7182	1.5656	0.7182	1.3924	2.1800	0.6387	0.4587	0.4587
3	0.6086	2.1743	1.9354	1.6430	3.5724	0.4599	0.2799	0.8902
4	0.5158	2.6901	3.4828	1.9388	5.2154	0.3717	0.1917	1.2947
5	0.4371	3.1272	5.2312	2.2878	7.1542	0.3198	0.1398	1.6728
6	0.3704	3.4976	7.0834	2.6996	9.4423	0.2859	0.1059	2.0252
7	0.3139	3.8115	8.9670	3.1855	12.1415	0.2624	0.0824	2.3526
8	0.2660	4.0776	10.8292	3.7589	15.3270	0.2452	0.0652	2.6558
9	0.2255	4.3030	12.6329	4.4355	19.0859	0.2324	0.0524	2.9358
10	0.1911	4.4941	14.3525	5.2338	23.5213	0.2225	0.0425	3.1936
11	0.1619	4.6560	15.9716	6.1759	28.7551	0.2148	0.0348	3.4303
12	0.1372	4.7932	17.4811	7.2876	34.9311	0.2086	0.0286	3.6470
13	0.1163	4.9095	18.8765	8.5994	42.2187	0.2037	0.0237	3.8449
14	0.0985	5.0081	20.1576	10.1472	50.8180	0.1997	0.0197	4.0250
15	0.0835	5.0916	21.3269	11.9737	60.9653	0.1964	0.0164	4.1887
16	0.0708	5.1624	22.3885	14.1290	72.9390	0.1937	0.0137	4.3369
17	0.0600	5.2223	23.3482	16.6722	87.0680	0.1915	0.0115	4.4708
18	0.0508	5.2732	24.2123	19.6731	103.7403	0.1896	0.0096	4.5916
19	0.0431	5.3162	24.9877	23.2144	123.4135	0.1881	0.0081	4.7003
20	0.0365	5.3527	25.6813	27.3930	146.6280	0.1868	0.0068	4.7978
21	0.0309	5.3837	26.3000	32.3238	174.0210	0.1857	0.0057	4.8851
22	0.0262	5.4099	26.8506	38.1421	206.3448	0.1848	0.0048	4.9632
23	0.0222	5.4321	27.3394	45.0076	244.4868	0.1841	0.0041	5.0329
24	0.0188	5.4509	27.7725	53.1090	289.4944	0.1835	0.0035	5.0950
25	0.0159	5.4669	28.1555	62.6686	342.6035	0.1829	0.0029	5.1502
30	0.0070	5.5168	29.4864	143.3706	790.9480	0.1813	0.0013	5.3448
40	0.0013	5.5482	30.5269	750.3783	4,163.2130	0.1802	0.0002	5.5022
50	0.0003	5.5541	30.7856	3,927.3569	21,813.0937	0.1800		5.5428
60	0.0001	5.5553	30.8465	20,555.1400	114,189.6665	0.1800		5.5526
100		5.5556	30.8642	15,424,131.91	85,689,616.17	0.1800		5.5555

Industrial and Systems Engineering

Linear Programming

The general linear programming (LP) problem is:

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$x_1, \dots, x_n \geq 0$$

An LP problem is frequently reformulated by inserting non-negative slack and surplus variables. Although these variables usually have zero costs (depending on the application), they can have non-zero cost coefficients in the objective function. A slack variable is used with a "less than" inequality and transforms it into an equality. For example, the inequality $5x_1 + 3x_2 + 2x_3 \leq 5$ could be changed to $5x_1 + 3x_2 + 2x_3 + s_1 = 5$ if s_1 were chosen as a slack variable. The inequality $3x_1 + x_2 - 4x_3 \geq 10$ might be transformed into $3x_1 + x_2 - 4x_3 - s_2 = 10$ by the addition of the surplus variable s_2 . Computer printouts of the results of processing an LP usually include values for all slack and surplus variables, the dual prices, and the reduced costs for each variable.

Dual Linear Program

Associated with the above linear programming problem is another problem called the dual linear programming problem. If we take the previous problem and call it the primal problem, then in matrix form the primal and dual problems are respectively:

<u>Primal</u>	<u>Dual</u>
Maximize $Z = \mathbf{c}\mathbf{x}$	Minimize $W = \mathbf{y}\mathbf{b}$
Subject to: $\mathbf{A}\mathbf{x} \leq \mathbf{b}$	Subject to: $\mathbf{y}\mathbf{A} \geq \mathbf{c}$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

It is assumed that if \mathbf{A} is a matrix of size $[m \times n]$, then \mathbf{y} is a $[1 \times m]$ vector, \mathbf{c} is a $[1 \times n]$ vector, \mathbf{b} is an $[m \times 1]$ vector, and \mathbf{x} is an $[n \times 1]$ vector.

Network Optimization

Assume we have a graph $G(N, A)$ with a finite set of nodes N and a finite set of arcs A . Furthermore, let

$$N = \{1, 2, \dots, n\}$$

$$x_{ij} = \text{flow from node } i \text{ to node } j$$

$$c_{ij} = \text{cost per unit flow from } i \text{ to } j$$

$$u_{ij} = \text{capacity of arc } (i, j)$$

$$b_i = \text{net flow generated at node } i$$

We wish to minimize the total cost of sending the available supply through the network to satisfy the given demand. The minimal cost flow model is formulated as follows:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij}x_{ij}$$

subject to

$$\sum_{j=1}^n x_{ij} - \sum_{j=1}^n x_{ji} = b_i \text{ for each node } i \in N$$

and

$$0 \leq x_{ij} \leq u_{ij} \text{ for each arc } (i, j) \in A$$

The constraints on the nodes represent a conservation of flow relationship. The first summation represents total flow out of node i , and the second summation represents total flow into node i . The net difference generated at node i is equal to b_i .

Many models, such as shortest-path, maximal-flow, assignment and transportation models, can be reformulated as minimal-cost network flow models.

Process Capability

Actual Capability

$$PCR_k = C_{pk} = \min\left(\frac{\mu - LSL}{3\sigma}, \frac{USL - \mu}{3\sigma}\right)$$

Potential Capability (i.e., Centered Process)

$$PCR = C_p = \frac{USL - LSL}{6\sigma}$$

where

μ and σ are the process mean and standard deviation, respectively, and LSL and USL are the lower and upper specification limits, respectively.

Queueing Models

Definitions

- P_n = probability of n units in system
- L = expected number of units in the system
- L_q = expected number of units in the queue
- W = expected waiting time in system
- W_q = expected waiting time in queue
- λ = mean arrival rate (constant)
- $\tilde{\lambda}$ = effective arrival rate
- μ = mean service rate (constant)
- ρ = server utilization factor
- s = number of servers

Kendall notation for describing a queueing system:

$$A / B / s / M$$

- A = the arrival process
- B = the service time distribution
- s = the number of servers
- M = the total number of customers including those in service

Fundamental Relationships

- $L = \lambda W$
- $L_q = \lambda W_q$
- $W = W_q + 1/\mu$
- $\rho = \lambda/(s\mu)$

Single Server Models ($s = 1$)

Poisson Input—Exponential Service Time: $M = \infty$

$$P_0 = 1 - \lambda/\mu = 1 - \rho$$

$$P_n = (1 - \rho)\rho^n = P_0\rho^n$$

$$L = \rho/(1 - \rho) = \lambda/(\mu - \lambda)$$

$$L_q = \lambda^2/[\mu(\mu - \lambda)]$$

$$W = 1/[\mu(1 - \rho)] = 1/(\mu - \lambda)$$

$$W_q = W - 1/\mu = \lambda/[\mu(\mu - \lambda)]$$

Finite queue: $M < \infty$

$$\tilde{\lambda} = \lambda(1 - P_M)$$

$$P_0 = (1 - \rho)/(1 - \rho^{M+1})$$

$$P_n = [(1 - \rho)/(1 - \rho^{M+1})]\rho^n$$

$$L = \rho/(1 - \rho) - (M + 1)\rho^{M+1}/(1 - \rho^{M+1})$$

$$L_q = L - (1 - P_0)$$

$$W = L/\tilde{\lambda}$$

$$W = W_q + 1/\mu$$

Poisson Input—Arbitrary Service Time

Variance σ^2 is known. For constant service time, $\sigma^2 = 0$.

$$P_0 = 1 - \rho$$

$$L_q = (\lambda^2\sigma^2 + \rho^2)/[2(1 - \rho)]$$

$$L = \rho + L_q$$

$$W_q = L_q/\lambda$$

$$W = W_q + 1/\mu$$

Poisson Input—Erlang Service Times, $\sigma^2 = 1/(k\mu^2)$

$$L_q = [(1 + k)/(2k)][(\lambda^2)/(\mu(\mu - \lambda))]$$

$$= [\lambda^2/(k\mu^2) + \rho^2]/[2(1 - \rho)]$$

$$W_q = [(1 + k)/(2k)]\{\lambda/[\mu(\mu - \lambda)]\}$$

$$W = W_q + 1/\mu$$

Multiple Server Model ($s > 1$)

Poisson Input—Exponential Service Times

Because calculations for P_0 and L_q can be time consuming, the following table gives formulas for 1, 2, and 3 servers.

s	P_0	L_q
1	$1 - \rho$	$\rho^2/(1 - \rho)$
2	$(1 - \rho)/(1 + \rho)$	$2\rho^3/(1 - \rho^2)$
3	$\frac{2(1 - \rho)}{2 + 4\rho + 3\rho^2}$	$\frac{9\rho^4}{2 + 2\rho - \rho^2 - 3\rho^3}$

$$P_n = P_0(\lambda/\mu)^n/n! \quad 0 \leq n \leq s$$

$$P_n = P_0(\lambda/\mu)^n/(s!s^{n-s}) \quad n \geq s$$

$$W_q = L_q/\lambda$$

$$W = W_q + 1/\mu$$

$$L = L_q + \lambda/\mu$$

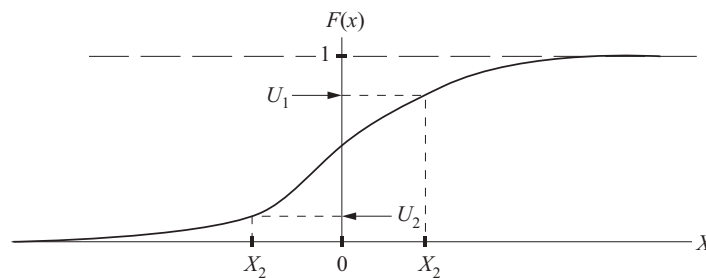
Simulation

1. Random Variate Generation

The linear congruential method of generating pseudo-random numbers U_i between 0 and 1 is obtained using $Z_n = (aZ_{n-1} + C) \pmod{m}$ where a , C , m , and Z_0 are given nonnegative integers and where $U_i = Z_i/m$. Two integers are equal (mod m) if their remainders are the same when divided by m .

2. Inverse Transform Method

If X is a continuous random variable with cumulative distribution function $F(x)$, and U_i is a random number between 0 and 1, then the value of X_i corresponding to U_i can be calculated by solving $U_i = F(x_i)$ for x_i . The solution obtained is $x_i = F^{-1}(U_i)$, where F^{-1} is the inverse function of $F(x)$.



Inverse Transform Method for Continuous Random Variables

Forecasting

Moving Average

$$\hat{d}_t = \frac{\sum_{i=1}^n d_{t-i}}{n}$$

where

\hat{d}_t = forecasted demand for period t

d_{t-i} = actual demand for i th period preceding t

n = number of time periods to include in the moving average

Exponentially Weighted Moving Average

$$\hat{d}_t = \alpha d_{t-1} + (1 - \alpha) \hat{d}_{t-1}$$

where

\hat{d}_t = forecasted demand for t

α = smoothing constant, $0 \leq \alpha \leq 1$

2ⁿ Factorial Experiments

$$E_i = \bar{Y}_{i2} - \bar{Y}_{i1}$$

$$E_{ij} = \frac{(\bar{Y}_{ij}^{22} - \bar{Y}_{ij}^{21}) - (\bar{Y}_{ij}^{12} - \bar{Y}_{ij}^{11})}{2}$$

where

Factors: X_1, X_2, \dots, X_n

Levels of each factor: 1, 2 (sometimes these levels are represented by the symbols – and +, respectively)

r = number of observations for each experimental condition (treatment)

E_i = estimate of the effect of factor X_i , $i = 1, 2, \dots, n$

E_{ij} = estimate of the effect of the interaction between factors X_i and X_j

\bar{Y}_{ik} = average response value for all $r2^{n-1}$ observations having X_i set at level k , $k = 1, 2$

\bar{Y}_{ij}^{km} = average response value for all $r2^{n-2}$ observations having X_i set at level k , $k = 1, 2$, and X_j set at level m , $m = 1, 2$.

Analysis of Variance for 2ⁿ Factorial Designs

Main Effects

Let E be the estimate of the effect of a given factor, let L be the orthogonal contrast belonging to this effect. It can be proved that

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^m a_{(c)} \bar{Y}^{(c)}$$

$$SS_L = \frac{rL^2}{2^n}$$

where

- m = number of experimental conditions ($m = 2^n$ for n factors)
- $a_{(c)}$ = -1 if the factor is set at its low level (Level 1) in experimental condition c
- $a_{(c)}$ = +1 if the factor is set at its high level (Level 2) in experimental condition c
- r = number of replications for each experimental condition
- $\bar{Y}_{(c)}$ = average response value for experimental condition c
- SS_L = sum of squares associated with the factor

Interaction Effects

Consider any group of two or more factors.

$a_{(c)} = +1$ if there is an even number (or zero) of factors in the group set at the low level (Level 1) in experimental condition $c = 1, 2, \dots, m$

$a_{(c)} = -1$ if there is an odd number of factors in the group set at the low level (Level 1) in experimental condition $c = 1, 2, \dots, m$

It can be proved that the interaction effect E for the factors in the group and the corresponding sum of squares SS_L can be determined as follows:

$$E = \frac{L}{2^{n-1}}$$

$$L = \sum_{c=1}^m a_{(c)} \bar{Y}_{(c)}$$

$$SS_L = \frac{rL^2}{2^n}$$

Sum of Squares of Random Error

The sum of the squares due to the random error can be computed as

$$SS_{\text{error}} = SS_{\text{total}} - \sum_i SS_i - \sum_i \sum_j SS_{ij} - \dots - SS_{12\dots n}$$

where SS_i is the sum of squares due to factor X_i , SS_{ij} is the sum of squares due to the interaction of factors X_i and X_j , and so on. The total sum of squares is equal to

$$SS_{\text{total}} = \sum_{c=1}^m \sum_{k=1}^r Y_{ck}^2 - \frac{T^2}{N}$$

where Y_{ck} is the k th observation taken for the c th experimental condition, $m = 2^n$, T is the grand total of all observations, and $N = r2^n$.

Reliability

If P_i is the probability that component i is functioning, a reliability function $R(P_1, P_2, \dots, P_n)$ represents the probability that a system consisting of n components will work.

For n independent components connected in series,

$$R(P_1, P_2, \dots, P_n) = \prod_{i=1}^n P_i$$

For n independent components connected in parallel,

$$R(P_1, P_2, \dots, P_n) = 1 - \prod_{i=1}^n (1 - P_i)$$

Learning Curves

The time to do the repetition N of a task is given by

$$T_N = KN^s$$

where

K = constant

s = $\ln(\text{learning rate, as a decimal})/\ln 2$; or, learning rate = 2^s

If N units are to be produced, the average time per unit is given by

$$T_{\text{avg}} = \frac{K}{N(1+s)} \left[(N+0.5)^{(1+s)} - 0.5^{(1+s)} \right]$$

Inventory Models

For instantaneous replenishment (with constant demand rate, known holding and ordering costs, and an infinite stockout cost), the economic order quantity is given by

$$EOQ = \sqrt{\frac{2AD}{h}}$$

where

A = cost to place one order

D = number of units used per year

h = holding cost per unit per year

Under the same conditions as above with a finite replenishment rate, the economic manufacturing quantity is given by

$$EMQ = \sqrt{\frac{2AD}{h(1-D/R)}}$$

where R = the replenishment rate

Facility Planning

Equipment Requirements

$$M_j = \sum_{i=1}^n \frac{P_{ij}T_{ij}}{C_{ij}}$$

where

M_j = number of machines of type j required per production period

P_{ij} = desired production rate for product i on machine j , measured in pieces per production period

T_{ij} = production time for product i on machine j , measured in hours per piece

C_{ij} = number of hours in the production period available for the production of product i on machine j

n = number of products

People Requirements

$$A_j = \sum_{i=1}^n \frac{P_{ij}T_{ij}}{C_{ij}}$$

where

A_j = number of crews required for assembly operation j

P_{ij} = desired production rate for product i and assembly operation j (pieces per day)

T_{ij} = standard time to perform operation j on product i (minutes per piece)

C_{ij} = number of minutes available per day for assembly operation j on product i

n = number of products

Standard Time Determination

$$ST = NT \times AF$$

where

NT = normal time

AF = allowance factor

Case 1: Allowances are based on the *job time*.

$$AF_{\text{job}} = 1 + A_{\text{job}}$$

A_{job} = allowance fraction (percentage/100) based on job time.

Case 2: Allowances are based on *workday*.

$$AF_{\text{time}} = 1/(1 - A_{\text{day}})$$

A_{day} = allowance fraction (percentage/100) based on workday.

Predetermined time systems are useful in cases where either (1) the task does not yet exist or (2) changes to a task are being designed and normal times have not yet been established for all elements of the new task or changed task. In such cases no opportunity exists to measure the element time. Unfortunately, there is no scientific basis for predicting element times without breaking them down into motion-level parts. A task consists of elements. An organization may develop its own database of normal element durations, and normal times for new or changed tasks may be predicted if the tasks consist entirely of elements whose normal times are already in the database. But new elements can be decomposed into motions, for which scientifically predetermined times exist in databases called MTM-1, MTM-2, and MTM-3. These databases and software to manipulate them are commercially available. To use one of them effectively requires about 50 hours of training.

Plant Location

The following is one formulation of a discrete plant location problem.

Minimize

$$z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}y_{ij} + \sum_{j=1}^n f_jx_j$$

subject to

$$\sum_{i=1}^m y_{ij} \leq mx_j, \quad j = 1, \dots, n$$

$$\sum_{j=1}^n y_{ij} = 1, \quad i = 1, \dots, m$$

$$y_{ij} \geq 0, \text{ for all } i, j$$

$$x_j = (0, 1), \text{ for all } j$$

Kennedy, W.J., and Daniel P. Rogers, *Review for the Professional Engineers' Examination in Industrial Engineering*, 2012.

where

m = number of customers

n = number of possible plant sites

y_{ij} = fraction or proportion of the demand of customer i which is satisfied by a plant located at site j ;
 $i = 1, \dots, m; j = 1, \dots, n$

x_j = 1, if a plant is located at site j

x_j = 0, otherwise

c_{ij} = cost of supplying the entire demand of customer i from a plant located at site j

f_j = fixed cost resulting from locating a plant at site j

Material Handling

Distances between two points (x_1, y_1) and (x_2, y_2) under different metrics:

Euclidean:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Rectilinear (or Manhattan):

$$D = |x_1 - x_2| + |y_1 - y_2|$$

Chebyshev (simultaneous x and y movement):

$$D = \max(|x_1 - x_2|, |y_1 - y_2|)$$

Line Balancing

$$N_{\min} = \left(OR \times \sum_i t_i / OT \right)$$

= theoretical minimum number of stations

$$\text{Idle Time/Station} = CT - ST$$

$$\text{Idle Time/Cycle} = \sum (CT - ST)$$

$$\text{Percent Idle Time} = \frac{\text{Idle Time/Cycle}}{N_{\text{actual}} \times CT} \times 100$$

where

CT = cycle time (time between units)

OT = operating time/period

OR = output rate/period

ST = station time (time to complete task at each station)

t_i = individual task times

N = number of stations

Job Sequencing

Two Work Centers—Johnson's Rule

1. Select the job with the shortest time, from the list of jobs, and its time at each work center.
2. If the shortest job time is the time at the first work center, schedule it first, otherwise schedule it last. Break ties arbitrarily.
3. Eliminate that job from consideration.
4. Repeat 1, 2, and 3 until all jobs have been scheduled.

Critical Path Method (CPM)

$$T = \sum_{(i,j) \in CP} d_{ij}$$

where

d_{ij} = duration of activity (i, j)

CP = critical path (longest path)

T = duration of project

PERT

$$\mu_{ij} = \frac{a_{ij} + 4b_{ij} + c_{ij}}{6}$$

$$\sigma_{ij} = \frac{c_{ij} - a_{ij}}{6}$$

$$\mu = \sum_{(i,j) \in CP} \mu_{ij}$$

$$\sigma^2 = \sum_{(i,j) \in CP} \sigma_{ij}^2$$

where

(a_{ij}, b_{ij}, c_{ij}) = (optimistic, most likely, pessimistic) durations for activity (i, j)

μ_{ij} = mean duration of activity (i, j)

σ_{ij} = standard deviation of the duration of activity (i, j)

μ = project mean duration

σ = standard deviation of project duration

Taylor Tool Life Formula

$$VT^n = C$$

where

V = speed in surface feet per minute

T = tool life in minutes

C, n = constants that depend on the material and on the tool

Work Sampling Formulas

$$D = Z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \quad \text{and} \quad R = Z_{\alpha/2} \sqrt{\frac{1-p}{pn}}$$

where

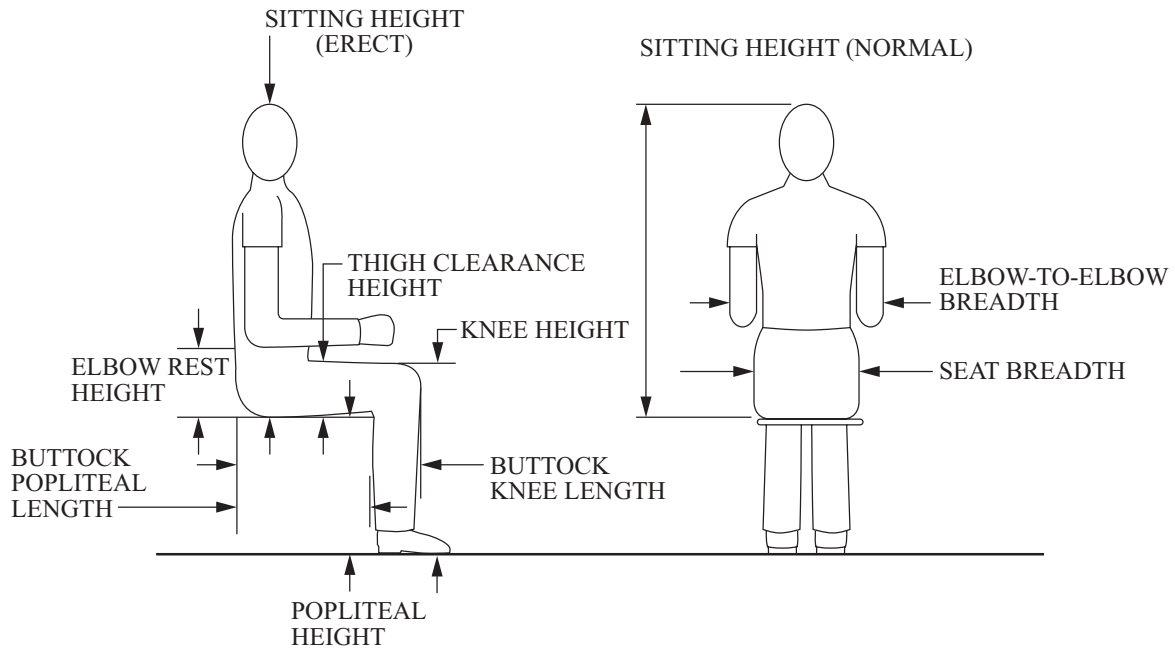
p = proportion of observed time in an activity

D = absolute error

R = relative error = D/p

n = sample size

ANTHROPOMETRIC MEASUREMENTS

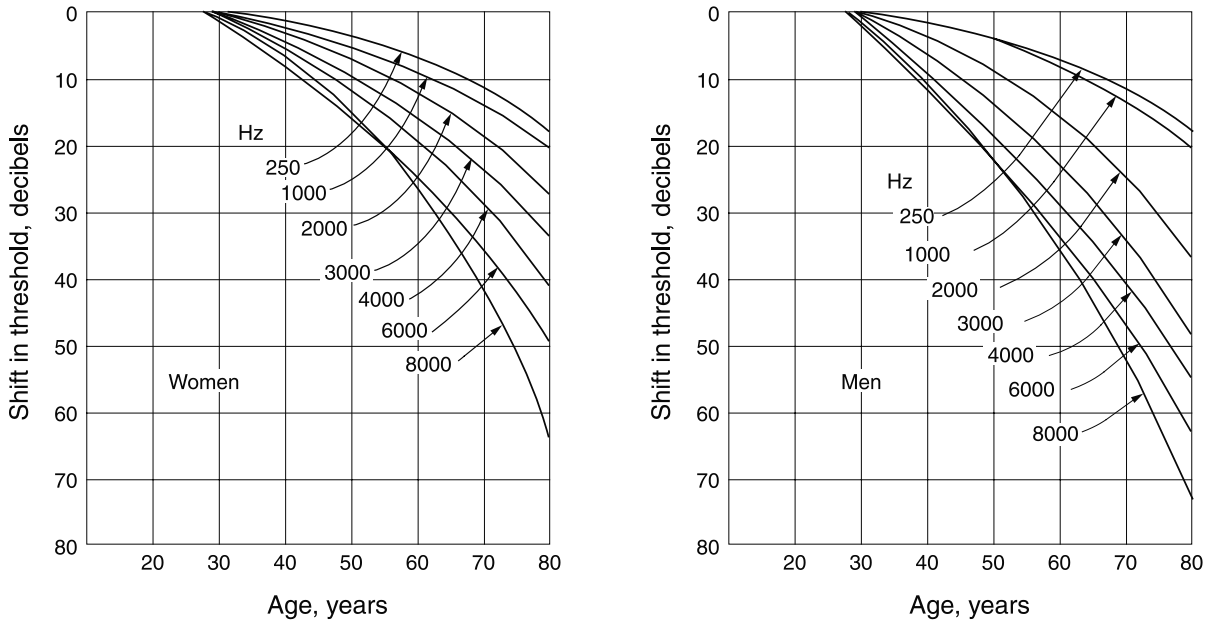


After Sanders and McCormick, *Human Factors In Engineering and Design*, McGraw-Hill, 1987.

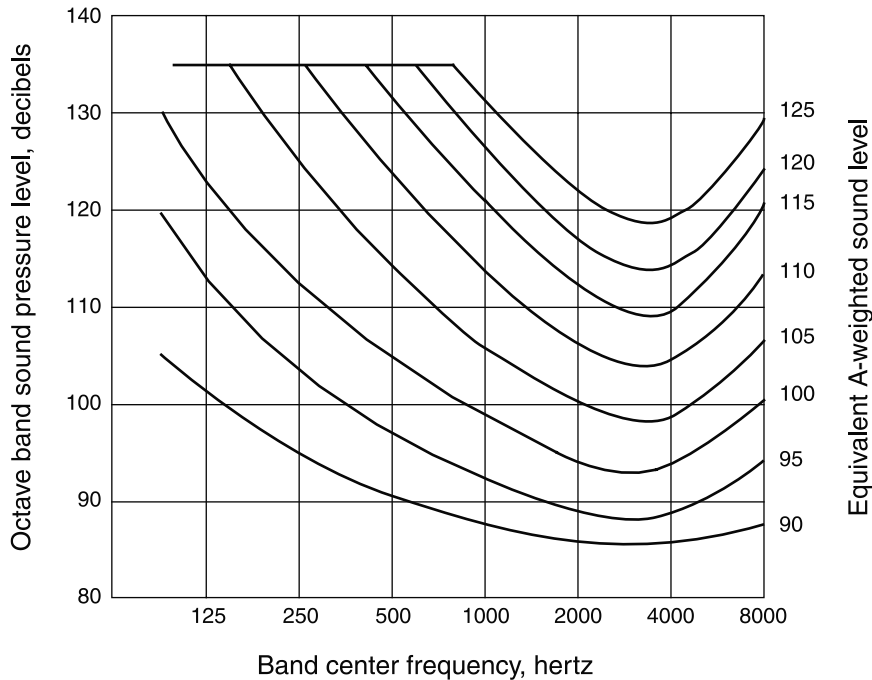
U.S. Civilian Body Dimensions, Female/Male, for Ages 20 to 60 Years (Centimeters)				
(See Anthropometric Measurements Figure)	Percentiles			
	5th	50th	95th	Std. Dev.
HEIGHTS				
Stature (height)	149.5 / 161.8	160.5 / 173.6	171.3 / 184.4	6.6 / 6.9
Eye height	138.3 / 151.1	148.9 / 162.4	159.3 / 172.7	6.4 / 6.6
Shoulder (acromion) height	121.1 / 132.3	131.1 / 142.8	141.9 / 152.4	6.1 / 6.1
Elbow height	93.6 / 100.0	101.2 / 109.9	108.8 / 119.0	4.6 / 5.8
Knuckle height	64.3 / 69.8	70.2 / 75.4	75.9 / 80.4	3.5 / 3.2
Height, sitting (erect)	78.6 / 84.2	85.0 / 90.6	90.7 / 96.7	3.5 / 3.7
Eye height, sitting	67.5 / 72.6	73.3 / 78.6	78.5 / 84.4	3.3 / 3.6
Shoulder height, sitting	49.2 / 52.7	55.7 / 59.4	61.7 / 65.8	3.8 / 4.0
Elbow rest height, sitting	18.1 / 19.0	23.3 / 24.3	28.1 / 29.4	2.9 / 3.0
Knee height, sitting	45.2 / 49.3	49.8 / 54.3	54.5 / 59.3	2.7 / 2.9
Popliteal height, sitting	35.5 / 39.2	39.8 / 44.2	44.3 / 48.8	2.6 / 2.8
Thigh clearance height	10.6 / 11.4	13.7 / 14.4	17.5 / 17.7	1.8 / 1.7
DEPTHS				
Chest depth	21.4 / 21.4	24.2 / 24.2	29.7 / 27.6	2.5 / 1.9
Elbow-fingertip distance	38.5 / 44.1	42.1 / 47.9	46.0 / 51.4	2.2 / 2.2
Buttock-knee length, sitting	51.8 / 54.0	56.9 / 59.4	62.5 / 64.2	3.1 / 3.0
Buttock-popliteal length, sitting	43.0 / 44.2	48.1 / 49.5	53.5 / 54.8	3.1 / 3.0
Forward reach, functional	64.0 / 76.3	71.0 / 82.5	79.0 / 88.3	4.5 / 5.0
BREADTHS				
Elbow-to-elbow breadth	31.5 / 35.0	38.4 / 41.7	49.1 / 50.6	5.4 / 4.6
Seat (hip) breadth, sitting	31.2 / 30.8	36.4 / 35.4	43.7 / 40.6	3.7 / 2.8
HEAD DIMENSIONS				
Head breadth	13.6 / 14.4	14.54 / 15.42	15.5 / 16.4	0.57 / 0.59
Head circumference	52.3 / 53.8	54.9 / 56.8	57.7 / 59.3	1.63 / 1.68
Interpupillary distance	5.1 / 5.5	5.83 / 6.20	6.5 / 6.8	0.4 / 0.39
HAND DIMENSIONS				
Hand length	16.4 / 17.6	17.95 / 19.05	19.8 / 20.6	1.04 / 0.93
Breadth, metacarpal	7.0 / 8.2	7.66 / 8.88	8.4 / 9.8	0.41 / 0.47
Circumference, metacarpal	16.9 / 19.9	18.36 / 21.55	19.9 / 23.5	0.89 / 1.09
Thickness, metacarpal III	2.5 / 2.4	2.77 / 2.76	3.1 / 3.1	0.18 / 0.21
Digit 1				
Breadth, interphalangeal	1.7 / 2.1	1.98 / 2.29	2.1 / 2.5	0.12 / 0.13
Crotch-tip length	4.7 / 5.1	5.36 / 5.88	6.1 / 6.6	0.44 / 0.45
Digit 2				
Breadth, distal joint	1.4 / 1.7	1.55 / 1.85	1.7 / 2.0	0.10 / 0.12
Crotch-tip length	6.1 / 6.8	6.88 / 7.52	7.8 / 8.2	0.52 / 0.46
Digit 3				
Breadth, distal joint	1.4 / 1.7	1.53 / 1.85	1.7 / 2.0	0.09 / 0.12
Crotch-tip length	7.0 / 7.8	7.77 / 8.53	8.7 / 9.5	0.51 / 0.51
Digit 4				
Breadth, distal joint	1.3 / 1.6	1.42 / 1.70	1.6 / 1.9	0.09 / 0.11
Crotch-tip length	6.5 / 7.4	7.29 / 7.99	8.2 / 8.9	0.53 / 0.47
Digit 5				
Breadth, distal joint	1.2 / 1.4	1.32 / 1.57	1.5 / 1.8	0.09 / 0.12
Crotch-tip length	4.8 / 5.4	5.44 / 6.08	6.2 / 6.99	0.44 / 0.47
FOOT DIMENSIONS				
Foot length	22.3 / 24.8	24.1 / 26.9	26.2 / 29.0	1.19 / 1.28
Foot breadth	8.1 / 9.0	8.84 / 9.79	9.7 / 10.7	0.50 / 0.53
Lateral malleolus height	5.8 / 6.2	6.78 / 7.03	7.8 / 8.0	0.59 / 0.54
Weight (kg)	46.2 / 56.2	61.1 / 74.0	89.9 / 97.1	13.8 / 12.6

Ergonomics—Hearing

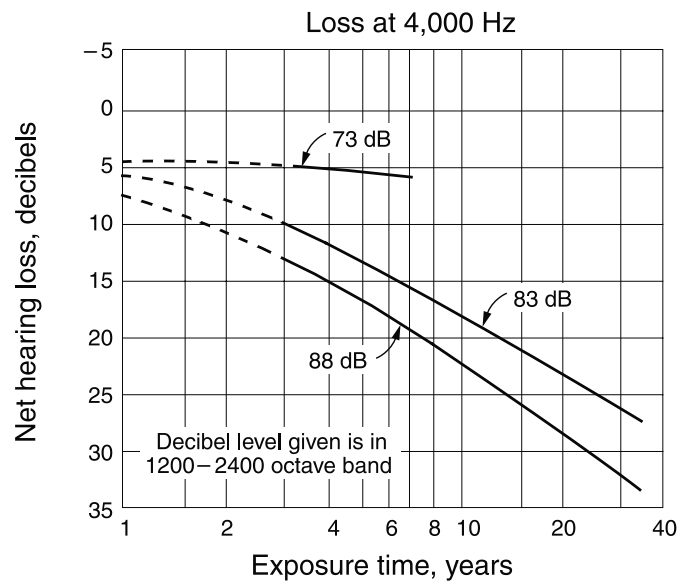
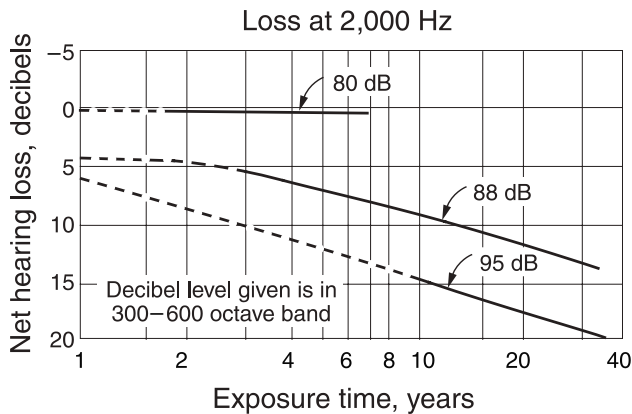
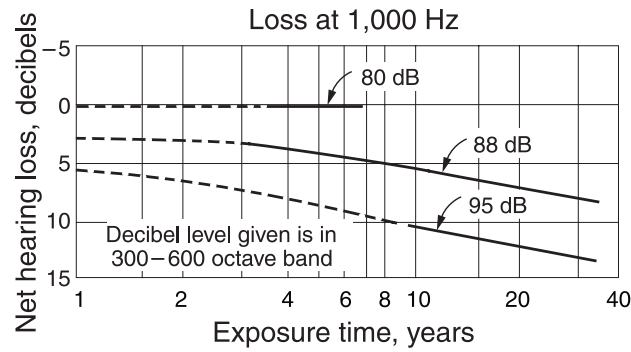
The average shifts with age of the threshold of hearing for pure tones of persons with "normal" hearing, using a 25-year-old group as a reference group.



Equivalent sound-level contours used in determining the A-weighted sound level on the basis of an octave-band analysis. The curve at the point of the highest penetration of the noise spectrum reflects the A-weighted sound level.



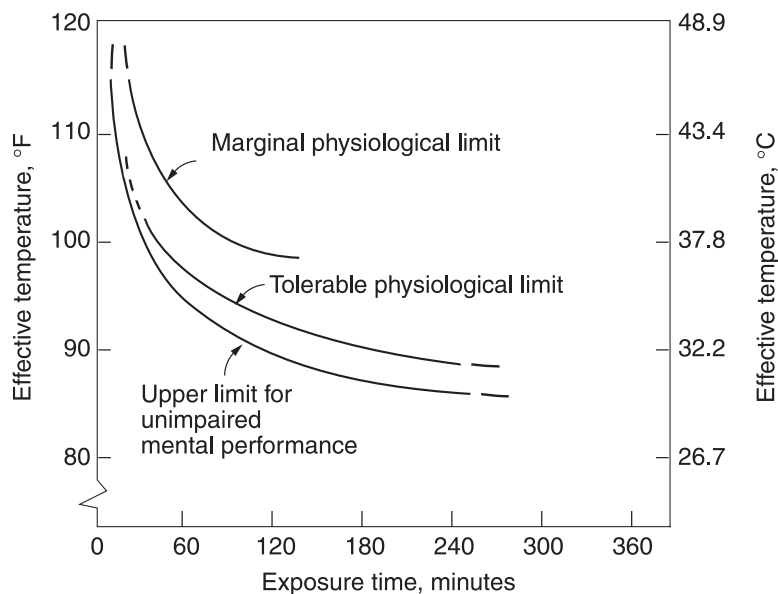
Estimated average trend curves for net hearing loss at 1,000, 2,000, and 4,000 Hz after continuous exposure to steady noise. Data are corrected for age, but not for temporary threshold shift. Dotted portions of curves represent extrapolation from available data.



"The Relations of Hearing Loss to Noise Exposure," Exploratory Subcommittee Z24-X-2 of the American Standards Association Z24 Special Committee on Acoustics, Vibration, and Mechanical Shock, sponsored by the Acoustical Society of America, American Standards Association, 1954, pp. 31-33.

Tentative upper limit of effective temperature (ET) for unimpaired mental performance as related to exposure time; data are based on an analysis of 15 studies. Comparative curves of tolerable and marginal physiological limits are also given.

Atmospheric Conditions



Effective temperature (ET) is the dry bulb temperature at 50% relative humidity, which results in the same physiological effect as the present conditions.