On the Primitive Classes of $K_{\ast}(BU)$

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ABSTRACT. We give a characterization of the primitive classes of $K_{\ast}(BU(n))$ in terms of its rational generators and use that to determine new primitive classes.

Introduction

The $BU$-ring spectrum $K$ determines a generalized homology theory $K_{\ast}$ with coefficient group

$$K_{n}(pt) = [S^{n}, K] = \pi_{n}(K)$$

It is well known that $K_{\ast}(K)$ can be regarded as a Hopf algebra over $\pi_{\ast}(K)$, and for each $X$ (space or spectrum) we can define a coaction map

$$\psi : K_{\ast}(X) \rightarrow K_{\ast}(K) \otimes_{\pi_{\ast}(K)} K_{\ast}(X)$$

which gives $K_{\ast}(K)$ the structure of a comodule over $K_{\ast}(K)$. The primitive submodule $PK_{\ast}(X)$ is defined by

$$PK_{\ast}(X) = \{ \alpha \in K_{\ast}(X) : \psi(\alpha) = 1 \otimes \alpha \}$$

In [1] we studied the case $X = MU(2)$ and gave a characteristic theorem which determines the primitive classes of $K_{\ast}(BU(2))$ in terms of its rational generators. In [2] the same case is studied (among other things) where $K_{\ast}(BU(2))$ is identified with a certain submodule of $Q[X, Y]$ whose homogeneous elements determine the primitive classes of $K_{\ast}(BU(2))$. In [3] the authors generalized the case of [2] to the space $BU[n]$ where they identify $K_{\ast}(BU(n))$ with a certain submodule of $Q[x_{1}, x_{2}, \ldots x_{n}]$ whose homogeneous elements again determine the primitive classes of $K_{\ast}(BU(n))$.

Although the above paper can be regarded as the leading and most comprehensive study concern $BU(n)$, but the primitive classes which were constructed there are all derived initially from $BU(2)$.
Here we generalize the results of [1] and give a characterization of the primitive classes of $K_*(BU(n))$ in terms of its rational generators and use that to determine new primitive classes of $K_*(BU(n))$ which are not derived from $K_*(BU(2))$.

§ 1. Notations

Let \{ $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ \} be the usual $\pi_*(K)$-basis of $K_*(CP^\infty)$. For each positive integer $n$ we define

\[ \Gamma_n = u^n(\beta_1)^n \quad b_n = u^n \beta_n \]

(1.1)

where $u \in \pi_2(K)$ is the usual generator and the product $(\beta_1)^n$ is induced by the tensor product: $CP^\infty \times CP^\infty \to CP^\infty$. Now using the result of [4], [5] one can prove (see [1] for details) that

\[ \Gamma_n = \sum_{r=1}^{n} r! S'_r b_r \]

(1.2)

were $S'_r$ is the sterling number of the second kind.

Let $i: CP^\infty \to BU$ be the canonical inclusion and denote the images of $\beta_n$, $\Gamma_n$ under $i_*$ also by $\beta_n$, $\Gamma_n$ respectively. The later classes of course can be multiplied in $K_*(BU)$ by using the Whitney sum maps: $BU(m) \times BU(n) \to BU(m + n)$. The following is a well known (see[6; p.47] or [7; 16.31]).

**Theorem 1.3**

(i) $K_*(BU(n))$ is free over $\pi_*(K)$ with a base consisting of the monomials

$\beta_{i_1} \beta_{i_2} \ldots \beta_{i_r}$

such that $i_1 > 0$, $i_2 > 0$, ..., $i_r > 0$, $0 \leq r \leq n$ (The monomial with $r = 0$ is interpreted as 1)

(ii) $K_*(MU(n))$ is free over $\pi_*(K)$ with a base consisting of the monomials

$\beta_{i_1} \beta_{i_2} \ldots \beta_{i_r}$

such that $i_1 > 0$, $i_2 > 0$, ..., $i_n > 0$.

(iii) $K_*(BU)$ is the polynomial algebra $\pi_*(K) [\beta_1, \beta_2, \ldots, \beta_n, \ldots]$.

**Remark 1.4**

If we replace the $\beta_i$’s by the $b_i$’s then all the statements given in the above theorem remain true.

§2. Primitivity in $K_*(BU)$

In this section we shall see that the classes $\Gamma_1$, $\Gamma_2$, ... play an important role in the determination of the primitive classes of $K_*(BU)$.

Consider the Hurewicz homomorphism $h^*_K: \pi_*(K) \to K_*(BU)$. It is easy to see that $\Gamma_1$ is in the image of $h^*_K$. But this is a natural homomorphism of Pontrjagin rings, hence
it follows that all the classes $\Gamma_1, \Gamma_2, \ldots$ are in the image of $h_K^s$ and hence

$$\text{Im } h_K^s \supseteq \mathbb{Z} [\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots]$$

Now since $K_\ast(\mathcal{B}U)$ is torsion-free, the map: $K_\ast(\mathcal{B}U) \to K_\ast(\mathcal{B}U) \otimes \mathbb{Q}$ is a monomorphism. By denoting the image of $\Gamma_n$ under this map also by $\Gamma_n$ we have

$$\text{Im } h_{KQ}^s \supseteq \mathbb{Q} [\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots]$$

where $h_{KQ}^s : \pi_\ast(\mathcal{B}U) \otimes \mathbb{Q} \to K_\ast(\mathcal{B}U) \otimes \mathbb{Q} = (KQ)_\ast(\mathcal{B}U)$.

In fact by a simple application of the Atiyah-Hirzebruch spectral sequence $E^2_{u, v} = \tilde{H}_u(\mathcal{B}U; \pi_v) \Rightarrow \pi_{u+v}(\mathcal{B}U)$

one can prove by comparing the ranks that (see [1] for details)

$$\text{Im } h_{KQ}^s = \mathbb{Q} [\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots]$$

(2.2)

Now since $\pi_\ast(K)$ is torsion-free $h_{KQ}^s$ maps $\pi_\ast(\mathcal{B}U) \otimes \mathbb{Q}$ isomorphically onto $P(KQ)_\ast(\mathcal{B}U)$. Hence we have proved the following:

**Proposition 2.3**

$$P(KQ)_\ast(\mathcal{B}U) = \mathbb{Q} [\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots]$$

The following result can be proved exactly the same as (2.3) or alternatively one can use the above result and the stable equivalence $\mathcal{B}U = \bigvee_{n=1}^{\infty} \mathcal{M}(n)$ of [8; Th. 1.4.2] to prove it.

**Proposition 2.4**

$P(KQ)_\ast(\mathcal{M}(n))$ is free over $\mathbb{Q}$ with a base consisting of all monomials $\Gamma_{i_1} \Gamma_{i_2} \ldots \Gamma_{i_n}$ such that $i_1 > 0, i_2 > 0, \ldots, i_n > 0$.

Note that such a monomial is not divisible in $K_\ast(\mathcal{M}(n))$, but more complicated expressions may be well be. In fact we have the following (see [1; 6.32]).

**Theorem 2.5**

An element $A$ in $K_\ast(\mathcal{M}(n))$ is primitive if and only if it can be written in the form

$$A = \sum_i \lambda_i \Gamma_{m_{i,1}} \Gamma_{m_{i,2}} \ldots \Gamma_{m_{i,n}} \quad \lambda_i \in \mathbb{Q}$$

such that when we rewrite it in terms of the $\pi_\ast(K)$-base $\{ b_{i_1}^1 b_{i_2}^2 \ldots b_{i_n}^n \}$, the induced formula has integral coefficients.

**Proof**

Since $K_\ast(\mathcal{M}(n))$ is torsion-free we have a monomorphism $\alpha : K_\ast(\mathcal{M}(n)) \to K_\ast(\mathcal{M}(n)) \otimes \mathbb{Q} = (KQ)_\ast(\mathcal{M}(n))$. Now let
Then by the above theorem $A$ is in $P(KQ)_s(MU(n))$. Now if $A$ also satisfies the condition of the theorem then it is in the image of $\alpha$ and so it represents an element of $PK_s(MU(n))$ as required.

Conversely if $A$ is in $PK_s(MU(n))$ then it is also in $P(KQ)_s(MU(n))$ where we identify $A$ with its image under the monomorphism $\alpha$. Hence by (2.4) we can write $A$ in the form

$$A = \sum_i \lambda_i \Gamma_{m_i,1} \Gamma_{m_i,2} ... \Gamma_{m_i,n} \lambda_i \in Q$$

Note that the condition of the theorem is satisfied since $A$ essentially is in $K_s(MU(n))$.

**Theorem 2.6**

An element $A$ in $K_s(MU(n))$ is primitive if and only if it can be written in the form

$$A = \sum_i \lambda_i \Gamma_{m_i,1} \Gamma_{m_i,2} ... \Gamma_{m_i,n} \lambda_i \in Q$$

such that $\lambda_1, \lambda_2, ... , \lambda_k$ are rational numbers satisfy the following condition

$$\frac{1}{n_1!n_2!...n_r!} \sum_i \lambda_i \left[ \sum_{\varphi} k_{\varphi(1)}^{m_i,1} k_{\varphi(2)}^{m_i,2} ... k_{\varphi(n)}^{m_i,n} \right]$$

is an integer for all $n$-tuples $(k_1, k_2, ... , k_n)$ of positive integers contains $r$- distinct elements repeated $n_1, n_2, ... n_r$ times, respectively, where $\varphi$ runs over all the permutations of $(1, 2, ... , n)$.

**Proof**

Suppose that $A$ is a primitive class of $K_s(MU(n))$. Then by the above theorem we can write it in the form

$$A = \sum_i \lambda_i \Gamma_{m_i,1} \Gamma_{m_i,2} ... \Gamma_{m_i,n} \lambda_i \in Q$$

such that when we write it in terms of the $\pi_s(K)$-base $\{b_1, b_2, ... b_n\}$, the induced formula has integral coefficients. Now by (1.2) we have

$$A = \sum_i \lambda_i \prod_{j=1}^n \left( \sum_{r_i,j=1}^{r_i} (r_i,j)! S_{m_i,j}^{r_i,j} b_{r_i,j} \right).$$

Let $a_{k_1, k_2, ... , k_n}$ be the coefficient of $b_{k_1} b_{k_2} ... b_{k_n}$. Then we have
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\[a(k_1, k_2, \ldots, k_n) = \frac{1}{n_1!n_2! \ldots n_r!} \sum_i \lambda_i \left\{ \sum_{\varphi} \prod_{j=1}^n (k_{\varphi(j)}! \cdot S_{m_{i,j}}^{k_{\varphi(j)}}) \right\} .\]

To complete the long and tedious proof of the theorem one use the induction on \( k = k_1 + k_2 + \ldots + k_n \) together with the formula (see [9; p. 226]).

\[r! S_n^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r-t)^n\]

See [1; 6.33] for the proof of the two dimensional case.

**Remark 2.7.** As we mentioned before the primitive classes of \( K^*(BU(n)) \) are represented in [3] by rational polynomials satisfy certain conditions. The primitive class \( A \) in the above theorem is corresponding to the rational polynomial \( f(x_1, \ldots, x_n) \) defined by

\[f(x_1, \ldots, x_n) = \frac{1}{n!} \sum_i \lambda_i \left( \sum_{\varphi} x_{\varphi(1)}^{m_{i,1}} x_{\varphi(2)}^{m_{i,2}} \ldots x_{\varphi(n)}^{m_{i,n}} \right) .\]

§3. Some Primitive Elements in \( K^*(BU) \)

Here we shall use theorem (2.6) to determine the primitive classes of \( K^*(MU(n)) \) of the form \( \lambda(\Gamma_m^{n-1} \Gamma_{m+n}s - G_m^n) \), where \( \lambda \in \mathbb{Q} \).

By the above theorem such a class is primitive if and only if the following expression

\[\lambda = \frac{(n-1)!}{n_1!n_2! \ldots n_r!} (k_1 k_2 \ldots k_n)^m \left[ \sum_{i=1}^n k_i^{ns} - n(k_1 k_2 \ldots k_n)^s \right] \]

is an integer for each \( n \)-tuple \( (k_1, k_2, \ldots, k_n) \) of positive integers containing \( r \) distinct elements repeated \( n_1, n_2, \ldots, n_r \) times, respectively.

**Notations 3.1.** Let

(i) \( X_{(k_1 k_2, \ldots, k_n)} = \left( \sum_{i=1}^n k_i^{ns} \right) - n(k_1 k_2 \ldots k_n)^s \),

(ii) \( X_k = X_{(k,1, \ldots, 1)} \)

It is easy to show that

**Proposition 3.2**
Next we recall the definition of a numerical function $m(t)$ defined on the positive integers. Let $v_p(k)$ be the exponent of the prime $p$ in $k$, so that $k = \prod p^{v_p(k)}$.

**Definition 3.3.** [10] If $t$ is a positive integer, we define $m(t)$ by

$$v_2(m(t)) = \begin{cases} 1 & \text{if } t \text{ odd} \\ 2 + v_2(t) & \text{if } t \text{ even} \end{cases}$$

for $p$ odd $v_p(m(t)) = \begin{cases} 1 \quad & \text{if } p-1 \text{ not divide } t \\ 1 + v_p(t) \quad & \text{if } p-1 \text{ divides } t \end{cases}$

Let $M_n(t)$ be the highest common factor of the expressions $k^n(k^t-1)$ where $k$ runs over the positive integers. One can prove the following

**Proposition 3.4.**

[10; p. 143] For each prime $p$ we have,

$$v_p(M_n(t)) = \text{Min} \{ n, v_p(m(t)) \}$$

In particular when $n$ is large enough we have $M_n(t) = m(t)$.

Returning to our case we want to find the divisibility in the expression

$$\frac{(n-1)!}{n_1!n_2!...n_r!} (k_1k_2...k_n)^m X(k_1,k_2,...,k_n) \quad (*)$$

From 3.2 (ii) it is easy to show that when $n$ is an odd prime we can choose $s$ (may take $s = n - 1$) such $k^mX_k$ is a multiplication of $M_m(n, s)$ where we define

$$v_p(M_m(n, s)) = \text{Min} \{ m, v_p(nm^2(s)) \} \quad (3.5)$$

Note that when $m$ is big enough we have $M_m(n, s) = nm^2(s)$.

Now $n$ is an odd prime. Hence $(n-1)! / n_1n_2!...n_r!$ is an integer. Therefore it follows from the above remark together with formula (i) of (3.2) that the expression (*) is divisible by $M_m(n,s)$. We have proved the following:

**Proposition 3.6.**

Let $p$ be an odd prime. For each positive integer $m$ there is a primitive class in $K_u (MU(p))$ of the form
Finally we want to mention that the primitive classes constructed here are only examples of the use of theorem (2.6) and we can form more of them using the same method. Also all of these classes can be constructed in the language of [3] using the same method (see remark (2.7)), but here the primitive classes are more recognizable.

References

الفصول الجذرية للنظم الجبرية

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بار (BU(n)) للفراغات (K*(BU(n))) المستخلص. الفصول الجذرية للنظم الجبرية (BU(n)) بدأت دراستها منذ عام 1980 م. إلا أن جميع الدراسات السابقة تركزت على الحالة الخاصة عندما 

\[ n = 2 \]

في هذا البحث نحدد بعض الفصول الجذرية لنظم تكون فيها 

\[ n \geq 3 \]