On the relationship between Laplace transform and new integral transform
"Tarig Transform"

Tarig. M. Elzaki and Salih M. Elzaki
Department of Mathematics, Sudan University of Science and Technology (www.sustech.edu).

ABSTRACT
In this paper we discuss some relationship between Laplace transform and the new integral transform called Tarig transform, we solve first and second order ordinary differential equations with constant and non-constant coefficients, using both transforms, and showing Tarig transform is closely connected with Laplace transform.

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F′(s) = s F(s)−f(0)

Then:

G(u) = \frac{1}{u} F\left(\frac{1}{u}\right)−\frac{1}{u} F\left(\frac{1}{u}\right)−f(0) = \frac{1}{u} G(u) \frac{1}{u} F\left(\frac{1}{u}\right)

The generalization to nth order derivative in (iii) can be proved by using mathematical induction.

**Theorem (3)**

Let \( G′(u) \) and \( F′(s) \) denote Tarig and Laplace transform of the definite integral of \( f(t) \), \( h(t) = \int_0^t f(\tau)\,d\tau \). Then:

\[ G′(u) = T\left[h(t)\right] = u^2 G(u) \]

**Proof:**

By definition of Laplace transform, \( F′(s) = L\left[h(t)\right] = \frac{F(s)}{s} \)

Hence

\[ G′(u) = \frac{1}{u} F′\left(\frac{1}{u}\right) = \frac{1}{u} u^2 F\left(\frac{1}{u}\right) = u^2 G(u) \]

**Theorem (4):**

Let \( G(u) \) is Trig transform of \( f(t) \) then:

\[ T\left[f(t)\right] = \frac{1}{2} \int_0^\infty u^2 \,dG(u) + u^2 G(u) \]

**Proof:**

By definition of Tarig transform we have:

\[ \frac{d}{dt} G(\alpha) = \frac{2}{\alpha} \int_0^{\alpha} u \,dG(u) + \frac{1}{\alpha} \int_0^{\alpha} u \,dG(u) = \frac{1}{\alpha} \int_0^{\alpha} u \,dG(u) + u \frac{d}{dt} G(\alpha) \]

Then:

\[ T\left[f(t)\right] = \frac{1}{2} \int_0^\infty u^2 \,dG(u) + u \frac{d}{dt} G(\alpha) \]

**Theorem (5):**

Let \( G(u) \) is Tarig transform of \( f(t) \) then:

(i) \[ T\left[f(t)\right] = \frac{1}{2} \int_0^\infty u \,dG(u) - \frac{f(0)}{u} \]

(ii) \[ T\left[f(t)\right] = \frac{1}{2} \int_0^\infty u \,dG(u) - \frac{f(0)}{u} \]

**Proof:**

(i) From theorem (4), we have:

\[ T\left[f(t)\right] = \frac{1}{2} \int_0^\infty u \,dG(u) - \frac{f(0)}{u} \]

The proof of (ii) is similar to the proof of (i).

**Theorem (6) (Convolution):**

Let \( f(t) \) and \( g(t) \) be in \( A \), having Laplace transform \( F(s) \) and \( G(s) \), and Tarig transform \( M(u) \) and \( N(u) \). Then:

\[ T\left[f(t) f(t)\right] = u M(u) N(u) \]

**Proof:**

First recall that Laplace transforms of \( f* g \) is given by

\[ L\left[f* g(t)\right] = F(s)G(s) \]

Now, since, by the duality relation (2) we have,

\[ T\left[f(t) g(t)\right] = \frac{1}{u} \int_0^\infty f(t) g(t) \,dt \]

Thus,

\[ M(u) = \frac{F\left(\frac{1}{u}\right)}{u}, \quad N(u) = G\left(\frac{1}{u}\right) \]

Trig transform of \( f(t) g(t) \) is obtained as follows:

\[ T\left[f(t) g(t)\right] = \frac{F\left(\frac{1}{u}\right)}{u} G\left(\frac{1}{u}\right) = u M(u) N(u) \]

**Example (1):**

Consider the first-order ordinary differential equation,

\[ \frac{dx}{dt} + px = f(t), \quad t > 0 \]

\[ x(0) = a \quad (4) \]

Where \( p \) and \( a \) are constants and \( f(t) \) is an external input function so that its Laplace and Tarig Transforms are exist. First Solution by Laplace Transform:

\[ \chi(s) = \frac{a}{s+p} + \frac{F(s)}{s+p} \]

Where that \( \chi(s) \) and \( F(s) \) are Laplace transform of \( x(t) \) and \( f(t) \). Then \( x(t) = a e^{-pt} + L^{-1}\left[F(s)\right] \quad Or \]

\[ x(t) = a e^{-pt} + \int_0^t f(t - \tau) e^{-p\tau} \,d\tau \]

In particular if \( f(t) = c \equiv \text{constant} \), then the Solution of (3) becomes:

\[ x(t) = \frac{c}{p} + \left[a - \frac{c}{p}\right] e^{-pt} \]

**Second Solution By Tarig Transform:**

Using Tarig transform of equation (3) we get

\[ \chi(u) = \frac{1}{u^2} x(0) + P X(u) = F(u) \]

Where \( \chi(u) \) and \( F(u) \) are Tarig transform of \( x(t) \) and \( f(t) \) then:

\[ X(u) = u^2 \frac{f(u)}{1 + u^2} + \frac{au}{1 + u^2} \]

The inverse Tarig transform leads to the solution in the form.

\[ x(t) = \frac{c}{p} + \left[a - \frac{c}{p}\right] e^{-pt} \]

\[ When \ f(t) = c \]

**Example (2):**

Consider the ordinary differential equation with variable coefficients (Bessel's equation).

\[ ty'' + y' + ty = 0, \quad y(0) = 1 \quad (5) \]

Solution by Laplace Transform:

\[ L\{ty''\} + L\{y'\} + L\{ty\} = 0 \quad and \]

\[ \frac{d}{dx}[xy' - sy(0) - y'(0)] + sy - y(0) \cdot \frac{dy}{dx} = 0 \]
\[ y = \frac{A}{\sqrt{1+s^2}}, \quad \text{where } Y \text{ is Laplace transform of } y \text{ inverting} \]

we find: \[ y(t) = AJ_0(t) \]

Solution by Tarig Transform:

Take Tarig transform of equation (5) we have,
\[
\frac{1}{2}u \frac{d}{du} \left[ G(u) - \frac{1}{u} y(0) - \frac{1}{2} \frac{d}{du} \left[ G(u) + \frac{1}{u} y(0) - \frac{1}{u} y'(0) \right] \right] \]

\[
\frac{G(u)}{u} = \frac{1}{u} y(0) + \frac{1}{2} u \frac{d}{du} G(u) \]

Where \( G(u) \) is Tarig transform of \( y \). Let \( y'(0) = c \), we have:

\[
\left( u + u^2 \right) G'(u) = \left( 1 - u^2 \right) G(u) \quad \text{And} \quad G'(u) = \frac{1}{u + u^2} = \frac{2u'}{1 + u'^4}.
\]

Integrating two sides we get: \( \ln G(u) = \ln \frac{Au}{\sqrt{1+u'^4}}, \) where \( A \) is a constant.

Inversion gives the formal solution: \( y(t) = AJ_0(t) \)

This is the same solution.

Example (3):

Consider the following linear integral differential equation.
\[
f''(t) = \delta(t) + \int_0^t f(\tau) \cos(t - \tau) d\tau, \quad f(0) = 1 \quad (6)
\]

Solution By Laplace Transform:

By taking Laplace transform of (6), we get.
\[
sF(s) - f(0) = 1 + \frac{s}{s^2 + 1}F(s) \quad \text{Or} \quad F(s) = \frac{2}{s} + \frac{2}{s^2 + 1}
\]

Apply the inverse Laplace transform to find the solution of (6) in the form:
\[
f(t) = 2 + t^2
\]

Solution by Tarig Transform:

By using Tarig Transform to eq(6)we get:
\[
\frac{G(u)}{u^2} - \frac{1}{u} = \frac{1}{u} + uG(u) \left[ \frac{u}{1+u^4} \right]
\]

And \( G(u) - 2u = \frac{u^4}{1+u^4} G(u) \quad \text{Or} \quad G(u) = 2u + 2u^5 \)

Inverting this equation we obtain the solution in the form:
\[
f(t) = 2 + t^2
\]

This is the same solution.

Conclusions:

Tarig transform is a convenient tool for solving differential equations in the time domain without the need for performing an inverse Tarig transform and the connection of Tarig transform with Laplace transform goes much deeper.

References

### Appendix Tarig Transform of Some Functions

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