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Boolean-Based Symbolic Analysis for the Reliability of Coherent Multi-State Systems of Heterogeneous Components

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Abstract. This paper is devoted to the Boolean-based analysis of non-repairable coherent multi-state systems with independent non-identical multi-state components. We adapt several binary concepts and tools such as probability-ready expressions, Boolean quotients, the Boole-Shannon expansion, and the Karnaugh map to the multi-state case. The paper utilizes algebraic techniques of multiple-valued logic to evaluate each of the multiple levels of the system output as a binary or propositional function of the system multi-valued inputs. The formula of each of these levels is then written as a probability–ready expression, thereby allowing its immediate conversion, on a one-to-one basis, into a probability or expected value. The symbolic reliability analysis of two small systems (which serve as standard gold examples of coherent multi-state systems) is completed successfully herein, yielding results that have been checked symbolically, and are also shown to agree numerically with those obtained earlier. The algebraic techniques used are supplemented by illustrative visualization via multi-valued Karnaugh maps. Emphasis is placed on the generalization of concepts of coherent binary systems to those of coherent multi-state ones, rather than innovating new unfamiliar stand-alone concepts for these latter systems.

Keywords: System reliability, Probability-ready expression, Multi-state system, Multiple-valued logic, Minimal upper vector, Maximal lower vector.

1. Introduction

The reliability literature deals mainly with binary or dichotomous systems, in which both a system and its components have two states (*i.e.*, either operational or failed). However, in many practical situations, there are multiple levels of system capacity or performance and/or different component performance levels and multiple component failure modes having different impacts on the system performance [1-5]. These systems are modeled as multi-state systems (*MSSs*), which might be coherent or non-coherent [6-16]. This paper deals with the prominent class of non-repairable coherent *MSSs* with independent non-identical multi-state components. The main contribution of the paper is to demonstrate that, similarly to coherent binary systems, coherent multi-state systems can be conveniently analyzed with the aid of switching-algebraic techniques and tools.

The literature abounds with standard research techniques for the reliability analysis of *MSSs* [17-34]. Most of these standard techniques rely on the utilization of discrete non-binary functions [35-37] or multiple-valued logic [38-56]. The main theme of this paper is that instead of tightening or narrowing the paradigms of discrete functions or multi-valued logic to fit the multi-state reliability problem, one could generalize or enlarge the switching-algebraic reliability analysis to suit the multi-state case. The starting point in our scheme pertains to reliability *per se*, and hence the adaptation to the multi-state case is straightforward. By contrast, the starting point in the standard analysis does not relate directly to reliability, or even to probability, and has to augment its course of action with some probability techniques, which might be lacking in efficiency.

This paper extends algebraic techniques and tools of switching algebra or binary logic to ones of multiple-valued logic, so as to evaluate each of the multiple levels of the system output as an individual binary or propositional function of the system multi-valued inputs. The formula of each of these levels is then written as a probability–ready expression, thereby allowing its immediate conversion, on a one-to-one basis, into a probability or expected value. The analysis will be seen to be particularly simple when the multi-state system is binary-imaged, i.e., when its success at each specific level is dependent only on component successes at the same level [8, 9, 16, 30-34]. The paper strives to provide a pedagogically-oriented treatment that establishes a clear and insightful interrelationship between binary modeling and MSS modeling by stressing that multi-valued concepts are natural and simple extensions of two-valued ones. Visual insight secured through the use of Karnaugh maps aids in the comprehension of coherent-system concepts, whether they are binary and multi state. A notable achievement for the multi-state case is the clarification of the subtle relation between a minimal upper vector (MUV) at a certain level and a prime implicant of success (minimal path) at that level, or the dual relation between a maximal lower vector (MLV) at a certain level and a prime implicant of failure (minimal cutsets) at that level. Many authors (see, e.g., [9, 15]) consider that the MUVs and MLVs play the role of (or are synonymous to) minimal paths and minimal cutsets, respectively. However, a minimal path constitutes all the upper vectors extending (inclusively) from a particular MUV to the all-highest vector, while a minimal cutset comprises all the lower vectors extending (inclusively) from the all-0 vector to a particular MLV.

The organization of the remainder of this paper is as follows. Section 2 presents important assumptions, notation and nomenclature. Section 3 introduces the concept of Boolean quotient in a multi-valued context. Section 4 extends the concept of a probability-ready expression (PRE) from the binary to the multi-state case. Section 5 provides a quick review of the concept of the Boole-Shannon expansion, again with an emphasis on its interpretation in a multi-valued sense. In Sections 6 and 7, the paper makes its main point through the multi-valued analysis of two specific (albeit standard) multi-state systems. Section 6 deals with a nonhomogeneous two-component system, while Section 7 handles a homogenous binary-imaged three-component system. Section 8 explores the issue of duality, which is rooted in the theory of switching and discrete functions, wherein it spreads to binary and multi-state reliability. Section 9 concludes the paper.

2. Assumptions, Notation and Nomenclature

2.1 Assumptions

- The model considered is one of a multi-state system with multistate components [1, 6], specified by the structure or success function $S(\boldsymbol{X})$ [15]

$$S: \{0, 1, \cdots, m_1\} \times \{0, 1, \cdots, m_2\} \times \ldots \times \{0, 1, \cdots, m_n\} \rightarrow \{0, 1, \cdots, M\}. \quad (1)$$

- The system is generally non-homogeneous, i.e., the number of system states $(M + 1)$ and the numbers of component states $(m_1 + 1), (m_2 + 1), \cdots, (m_n + 1)$ might differ. When these numbers have a common value, the system reduces to a homogeneous one.

- The system is a non-repairable one with statistically independent non-identical (heterogeneous) components.
- The system is a coherent one enjoying the properties of causality, monotonicity, and component relevancy [1, 2, 4, 31-34].

The system is not necessarily binary-imaged or dominant [16].

2.2 Notation

Symbol	Description
X_k	A multivalued input variable representing component k $(1 \le k \le n)$, where $X_k \in \{0, 1, \dots, m_k\}$, and $m_k \ge 1$ is the highest value of X_k.
$X_k\{j\}$	A binary variable representing instant j of X_k $X_k\{j\} = \{X_k = j\},$ i.e., $X_k\{j\} = 1$ if $X_k = j$ and $X_k\{j\} = 0$ if $X_k \neq j$. The instances $X_k\{j\}$ for $\{0 \le j \le m_k\}$ form an orthonormal set, namely, for $\{1 \le k \le n\}$ $\bigvee_{j=0}^{m_k} X_k\{j\} = 1,$ (2a) $X_k(j_1)\, X_k(j_2) = 0$ for $j_1 \neq j_2$. (2b) Orthonormality is very useful in constructing inverses or complements. The complement of the union of certain instances is the union of the complementary instances. In particular, the complement of $X_k\{\ge j\} = X_k\{j, j+1, \dots, m_k\}$ is $X_k\{< j\} = X_k\{0, 1, \dots, j-1\}$.
$X_k\{\ge j\}$	An upper value of X_k $\{0 \le j \le m_k\}$: $X_k\{\ge j\} = X_k\{j, j+1, \dots, m_k\} = \bigvee_{i=j}^{m_k} X_k\{i\} = X_k\{j\} \vee X_k\{j+1\} \vee \dots \vee X_k\{m_k\}.$ (3) The value $X_k\{\ge 0\}$ is identically 1. The set $X_k\{\ge j\}$ for $\{1 \le j \le m_k\}$ is neither independent nor disjoint, and hence it is difficult to be handled mathematically, but it is very convenient for translating the verbal or map/tabular description of a coherent component into a mathematical form when viewing component success at level j. The complement of $X_k\{\ge j\}$ is $X_k\{< j\} = X\{0, 1, \dots, j-1\} = X_k\{0\} \vee X_k\{1\} \dots \vee X_k\{j-1\} = X_k\{k \le (j-1)\}.$ (4)
$X_k\{\le j\}$	A lower value of X_k $\{0 \le j \le m_k\}$: $X_k\{\le j\} = X_k\{0, 1, \dots, j-1, j\} = \bigvee_{i=0}^{j} X_k\{i\} = X_k\{0\} \vee X_k\{1\} \dots \vee X_k\{j-1\} \vee X_k\{j\}.$ (5) The value $X_k\{\le m_k\}$ is identically 1. The set $X_k\{\le j\}$ for $\{0 \le j \le (m_k - 1)\}$ is neither independent nor disjoint, and hence it is not convenient for mathematical manipulation though it is suitable for expressing component failure at level $(j+1)$. Instances, upper values and lower values are related by $X_k\{j\} = X_k\{\ge j\}\, X_k\{< (j+1)\} = X_k\{\ge j\}\, \bar{X}_k\{\ge (j+1)\} = X_k\{\le j\}\, X_k\{> (j-1)\}$ $= X_k\{\le j\}\, \bar{X}_k\{\le (j-1)\}.$ (6)

S	A multivalued output variable representing the system, where $$S \in \{0, 1, \dots, M\}, \quad (7)$$ and $M \geq 1$ is the highest value attained by the system. The system is called homogeneous if $M = m_1 = m_2 = \cdots = m_n$. The function $S(\boldsymbol{X})$ is usually called the system success or the structure function. It is conveniently represented by a Multi-Valued Karnaugh Map (MVKM) [30-34, 48, 57]. Its complement $\bar{S}(\boldsymbol{X})$ is called system failure, and is also a multivalued variable of $(M+1)$ values. The sum $(S(\boldsymbol{X}) + \bar{S}(\boldsymbol{X}))$ is identically equal to M.
$S\{j\}$	A binary variable representing instant j of S $$S\{j\} = \{S(\boldsymbol{X}) = j\}, \quad (8)$$ i.e., $S\{j\} = 1$ if $S(\boldsymbol{X}) = j$, and $S(\boldsymbol{X}) = 0$ if $S(\boldsymbol{X}) \neq j$. The instances $S\{j\}$ for $\{0 \leq j \leq M\}$ form an orthonormal set, i.e. $$\vee_{j=0}^{M} S\{j\} = 1, \quad (9)$$ $$S\{j_1\}\, S\{j_2\} = 0 \;\text{ for }\; j_1 \neq j_2, \quad (10)$$ which means that one, and only one, of the $(M+1)$ instances of S has the value 1, while the other instances are all $0's$.
$S\{\geq j\}$	An upper value of S $$S\{\geq j\} = S\{j, j+1, \dots, M\} = \bigvee_{i=j}^{M} S\{i\}. \quad (11)$$
$S\{\leq j\}$	A lower value of S $$S\{\leq j\} = S\{0, 1, \dots, j\} = \bigvee_{i=0}^{j} S\{i\}. \quad (12)$$ Instances, upper values and lower values of S are related by $$S\{j\} = S\{\geq j\}\, S\{< (j+1)\} = S\{\geq j\}\, \bar{S}\{\geq (j+1)\} = S\{\leq j\}\, S\{> (j-1)\}. \quad (13)$$

2.3 Nomenclature

2.3.1 A vector X:

- A specific value of the input arguments $\boldsymbol{X} = [X_1 \quad X_2 \cdots X_n]^T$ of the multi-valued structure function S;
- A particular cell of the MV Karnaugh map of S or the binary Karnaugh map of any of its instances $S\{j\}$, upper values $S\{\geq j\}$ or lower values $S\{\leq j\}$.

2.3.2 An upper vector for level $j > 0$

- A particular value of $\boldsymbol{X}$ such that $S(\boldsymbol{X}) \geq j$, $\{j = 1, 2, \dots, M\}$;
- A true vector for $S\{\geq j\}$, $i.e.$, a vector such that $S\{\geq j\} = 1, \{j = 1, 2, \dots, M\}$;
- A map cell for system success at level $j, \{j = 1, 2, \dots, M\}$.

2.3.3. A minimal upper vector (MUV) at level $j > 0$, denoted θ_{ji}

An upper vector $\boldsymbol{X}$ for level j such that $S(\boldsymbol{Y}) < j$, $\{j = 1, 2, \dots, \boldsymbol{M}\}$ for any vector $\boldsymbol{Y} <$

$\boldsymbol{X}$. Such a vector is a member of the set of MUVs at level $j > 0$, denoted by $\theta(j)$.

2.3.4 An upper prime implicant at level $j > 0$, denoted P_{ji}

- The set of upper vectors $\boldsymbol{X}$ for level $j\,\{1 \le j \le M\}$ such that

  $\boldsymbol{\theta}_{ji} \le \boldsymbol{X} \le \boldsymbol{U}$.

- The loop in the $S\{\ge j\}$ map whose cells are not lower than $\boldsymbol{\theta}_{ji}$;

$$P_{ji} = \bigwedge_{k=1}^{n} X_k\{\ge \theta_{ji}(k)\}, \quad (14)$$

$$S\{\ge j\} = \bigvee_{i} P_{ji}. \quad (15)$$

2.3.5 The all-highest vector U

The vector where each input argument attains its highest value

$$\boldsymbol{U} = [m_1 \quad m_2 \quad \cdots \quad m_n]^T.$$

The vector belongs to upper prime implicants at all levels, i.e. to P_{ji} for all i and all $j > 0$. Due to causality, the structure function must attain its maximum when $\boldsymbol{X} = \boldsymbol{U}, i.e.\ S(\boldsymbol{U}) = M$.

2.3.6 A lower vector for level $j < M$

- A particular value of $\boldsymbol{X}$ such that $S(\boldsymbol{X}) \le j, \{j = 0, 1, \dots, (M-1)\}$;
- A true vector for $S\{\le j\}, i.e.$, a vector such that $S\{\le j\} = 1, \{j = 0, 1, \dots, (M-1)\}$;
- A false vector for $S\{> j\} = S\{\ge (j+1)\}, j = 0, 1, \dots, (M-1)\}$;
- A map cell for system failure at level $j\,\{j = 1, 2, \dots, M\}$.

2.3.7 A maximal lower vector (MLV) at level $j < M$, denoted σ_{ji}

A lower vector $\boldsymbol{X}$ for level j such that $S\{\boldsymbol{Y}\} > j$, $\{j = 1, 2, \dots, M\}$ for any vector $\boldsymbol{Y} > \boldsymbol{X}$. Such a vector is a member of the set of MLVs at level $j < M$, denoted by $\boldsymbol{\sigma}(j)$.

2.3.8 A lower prime implicant at level $j < M$, denoted Q_{ji}

- The set of lower vectors $\boldsymbol{X}$ for level $j\,\{0 \le j \le (M-1)\}$ such that

  $\boldsymbol{L} \le \boldsymbol{X} \le \boldsymbol{\sigma}_{ji}$.

- The loop in the $S\{\le j\}$ map whose cells are not higher than $\boldsymbol{\sigma}_{ji}$;

$$Q_{ji} = \bigwedge_{k=1}^{n} X_k\{\le \sigma_{ji}(k)\}, \quad (16)$$

$$S\{\le j\} = \bigvee_{i} Q_{ji}. \quad (17)$$

2.3.9 The all-lowest vector L

The vector where each input argument attains its lowest value

$$\boldsymbol{L} = [0 \quad 0 \quad \cdots \quad 0]^T \quad (18)$$

This vector belongs to all lower prime implicants at all levels, i.e. to Q_{ji} for all i and all $j\ \{j < M\}$. Due to causality, the structure function attains its minimum when $\boldsymbol{X} = \boldsymbol{L}, i.e., S(\boldsymbol{L}) = 0$.

2.3.10 The expected value of a certain instance $S\{j\}$

The expected value of a certain instance $S\{j\}$ of $S, \{j = 0, 1, \dots, M\}$

$$E\{S\{j\}\} = E\{S\{\ge j\}\} - E\{S\{\ge (j+1)\}\} = E\{S\{\le j\}\} - E\{S\{\le (j-1)\}\}, \quad (19a)$$

where

$$E\{S\{\ge (M+1)\}\} = 0, \quad (19b)$$

$$E\{S\{\le (-1)\}\} = 0, \quad (19c)$$

$$E\{S\{\ge 0\}\} = 1, \quad (19d)$$

$$E\{S\{\le M\}\} = 1. \quad (19e)$$

The availability of two formulas for $E\{S\{j\}\}$ allows us to select the formula that is better in some sense (easier to derive, more compact to express, etc.). Otherwise, we might evaluate both formulas and check that they agree. Both sets of formulas $\{(19a), (19b), (19d)\}$ and formulas $\{(19a), (19c), (19e)\}$ confirm the arithmetic normality property

$$\sum_{j=0}^{M} E\{S\{j\}\} = 1. \quad (20)$$

2.3.11 The probability transform

The expectation $E\{..\}$ of any logic expression (binary or multi-valued) might be obtained through a probability-transform operation [58, 59]. An expression for $E\{S\}$ is a multi-affine function in its arguments (an algebraic function depicting a straight line relation in each of the arguments), and this expression has the same "truth table" as that of the logic function S [59]. Figure 1 illustrates the probability-transform operation for a system of two three-valued components. Despite the different mathematical natures of S and $E\{S\}$, they are both of a multi-affine structure, and they have 'truth tables' of exactly the same entries.

2.3.12 Various Relations between Two Component State Vectors X and Y

- A vector $\boldsymbol{X}$ is larger than another vector $\boldsymbol{Y}$ (denoted $\boldsymbol{X} > \boldsymbol{Y}$) if every element of $\boldsymbol{X}$ is at least as large as the corresponding element of $\boldsymbol{Y}$, and at least one element of $\boldsymbol{X}$ is larger than the corresponding element of $\boldsymbol{Y}$. If $\boldsymbol{X} > \boldsymbol{Y}$ then certainly $S(\boldsymbol{X}) \geq S(\boldsymbol{Y})$, and occasionally $S(\boldsymbol{X}) > S(\boldsymbol{Y})$**,** as a result of coherence.
- A vector $\boldsymbol{X}$ is equivalent to another vector $\boldsymbol{Y}$ (denoted $\boldsymbol{X} \leftrightarrow \boldsymbol{Y}$) if $S(\boldsymbol{X}) = S(\boldsymbol{Y})$, i.e., both are equal to $j, \{j = 0,1,2, \dots, M\}$. Therefore, the set of input vectors $\boldsymbol{X}$ is partitioned into $(M + 1)$ equivalence classes.
- A vector $\boldsymbol{X}$ dominates another vector $\boldsymbol{Y}$ if it is larger than it ($\boldsymbol{X} > \boldsymbol{Y}$), or it is larger than a vector $\boldsymbol{Z}$ in the same equivalence class as $\boldsymbol{Y}$ ($(\boldsymbol{X} > \boldsymbol{Z}) \wedge (S(\boldsymbol{Z}) = S(\boldsymbol{Y}))$ [16].

2.3.13 A Binary-Imaged Multi-State System

A binary-imaged multi-state system is a system whose success at level j is a function only of component successes at the same level ($S\{\geq j\}$ is a function of $\boldsymbol{X}\{\geq j\}$ only), or equivalently, it is a system whose failure at level j is a function only of component failures at the same level ($S\{\leq (j-1)\}$ is a function of $\boldsymbol{X}\{\leq (j-1)\}$ only) [34]. For a binary-imaged system, elements of the set of MUVs $\theta(j)$ are vectors of j or 0 components only, and elements of the set of MLVs $\sigma(j)$ are vectors of j or M components only [34]. Figure 2 shows Multi-valued Karnaugh maps (MVKMs) representing the structure functions of three small coherent three-state systems of two three-valued components, the first of which is binary imaged, while the remaining two are not binary imaged.

2.3.14 A Dominant Multi-State System

A dominant multi-state system is a coherent multi-state system, in which $S(\boldsymbol{X}) > S(\boldsymbol{Y})$ implies vector $\boldsymbol{X}$ dominates vector $\boldsymbol{Y}$. In a dominant system, every vector of state $j > 0$ must be larger than at least one vector of a smaller state value. A non-dominant system cannot be binary imaged [16]. Figure 2 shows (a) a system that is both dominant and binary imaged, (b) a dominant system that is not binary imaged, and (c) a system that is non-dominant and hence non binary imaged. Note that the system in Fig. 2(c) is non-dominant since its vector (2, 0) of state 2 is not larger any vector of state 1.

2.3.15 Multi-State Interpretation of Binary Systems

For a binary system ($M = 1$), there is a single level other than level 0, namely level 1. In this case, success at level 1 is $S\{\geq 1\} = S\{1\}$, while failure at level 1 is $S\{< 1\} = S\{0\}$. In the binary case, there is no need to refer to level 1 since it is the only non-zero level and is implicitly understood by default, and we simply refer to system success S and system failure $\bar{S}$ instead of $S\{1\}$ and $S\{0\}$.

3. Boolean Quotients

The concept of a Boolean quotient is an important switching-algebraic concept that can be conveniently viewed in a multi-valued context [31]. Given a two-valued Boolean function (a switching function) f and a term t, the Boolean quotient of f with respect to t, denoted by (f/t), is defined to be the function formed from f by imposing the constraint $\{t = 1\}$ explicitly [59, 60], namely

$$f/t = [f]_{t=1}, \tag{21}$$

The Boolean quotient is also known as a ratio, a sub-function, or a restriction. Brown [60] and Rushdi & Rushdi [59] list several useful properties of Boolean quotients. In the multi-valued context, the term t is a product (ANDing) of literals. Each of the multi-valued variables is either absent or present in the form of a particular literal, which might be a single instance or the ORing of several instances [31].

A fundamental property of the Boolean quotient states that a term ANDed with a function is equal to the term ANDed with the Boolean quotient of the function with respect to the term, namely.

$$t \wedge f \ = t \ \wedge \ (f/t\,). \tag{22}$$

If the term t is a factor of the function f (*i.e.*, $= t \ \wedge \ g$, $t \wedge f \ = f$), then (22) takes the simpler form

$$f \ = t \ \wedge \ (f/t\,). \tag{23}$$

In this paper, we denote a Boolean quotient by an inclined slash (f/t). However, it is possible to denote it by a vertical bar $(f|t)$ to stress the equivalent meaning (borrowed from conditional probability) of f conditioned by t or f given t [59].

4. Probability-Ready Expressions

The concept of a probability-ready expression (RRE) is well-known in the two-valued logical domain [59, 61-66], and it is still valid for the multi-valued logical domain [30-34]. A Probability-Ready Expression is a random expression that can be directly transformed, on a one-to-one basis, to its statistical expectation (its probability of being equal to 1) by replacing all logic variables by their statistical expectations, and also replacing logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts. A logic expression is a PRE if:

a) all *ORed* products (terms formed by ANDing) are *disjoint* (*mutually exclusive*), and

b) all *ANDed* sums (alterms formed via ORing) are *statistically independent*.

Condition (a) is satisfied if for every pair of ORed terms, there is at least a single opposition, i.e., there is at least one variable that appears with a certain set of instances in one term and appears with a complementary set of instances in the other. Condition (b) is satisfied if for every pair of ANDed alterms (sums of disjunctions of literals), one alterm involves variables describing a certain set of components, while the other alterm depends on variables describing a set of different components (under the assumption of independence of components) [31, 59, 61, 62, 66].

While there are many methods to introduce characteristic (a) of orthogonality (disjointness) into a Boolean expression [67-74], there is no way to induce characteristic (b) of statistical independence. The best that one can do is to observe statistical independence when it exists, and then be careful not to destroy or spoil it and take advantage of it. Since one has the freedom of handling a problem from a success or a failure perspective, a choice should be made as to which of the two perspectives can more readily produce a PRE form. It is better to look at success for a system of no or poor redundancy (a series or almost-series system), and to view failure for a system of full or significant redundancy (a parallel or almost- parallel system) [59, 61-66].

The introduction of orthogonality might be achieved as follows. If neither of the two terms A and B in the sum $(A \vee B)$ subsumes the other ($A \vee B \neq A$ and $A \vee B \neq B$) and the two terms are not disjoint ($A \wedge B \neq 0$), then B can be disjointed with A by factoring out any common factor (using (23)) and then applying the *Reflection Law*, namely

$$A \vee B = C\left(\left(\frac{A}{C}\right) \vee \left(\frac{B}{C}\right)\right) = C\left(\left(\frac{A}{C}\right) \vee \overline{\left(\frac{A}{C}\right)}\left(\frac{B}{C}\right)\right) = A \vee \overline{\left(\frac{A}{C}\right)} B. \tag{24}$$

In (24), the symbol C denotes the common factor of $A \; and \; B$, and the Boolean quotient (A/C) might be viewed as the term A with its part common with B removed. Note that (24) leaves the term A intact and replaces the term B by an expression that is disjoint with A. The quotient (A/C) is a product of e entities Y_k $(1 \leq k \leq e)$, so that $\overline{(A/C)}$ might be expressed via *De Morgan's Law* as a disjunction of the form

$$\overline{(A/C)} = \bigvee_{k=1}^{e} \bar{Y}_k. \tag{25}$$

Note that each Y_K stands for a disjunction of certain instances of some variable $X_{i(k)}$ *and hence* $\bar{Y}_k$ is a disjunction of the complementary instances of the same variable. If we combine (24) with (25), we realize that the term B is replaced by e terms $(e \geq 1)$, which are each disjoint with the term A, but are not necessarily disjoint among themselves. Therefore, we replace the *De Morgan's Law* in (25) by a disjoint version of it [59] namely

$$\begin{aligned}\overline{(A/C)} &= \bar{Y}_1 \vee Y_1\bar{Y}_2 \vee Y_1Y_2\bar{Y}_3 \vee \ldots \vee Y_1Y_2 \ldots Y_{e-1}\bar{Y}_e \\ &= \bar{Y}_1 \vee Y_1(\bar{Y}_2 \vee Y_2(\bar{Y}_3 \vee \ldots \ldots \vee (\bar{Y}_{e-1} \vee Y_{e-1}\bar{Y}_e) \ldots))\end{aligned} \tag{26}$$

When (26) is combined with (24), the first term A still remains intact, while the second term B is replaced by e terms which are each disjoint with A and are also disjoint among themselves. This means that one has a choice of either disjointing B with A in $A \vee B$, or disjointing A with B in $B \vee A$. The usual practice that is likely to yield good results is to order the terms in a given disjunction so that those with fewer literals appear earlier.

Rushdi [31] presented a simple example of the procedure above by considering the following expression

$$S\{0\} = X_1\{0\} \vee X_2\{0\} \vee X_3\{0\} \vee X_4\{0\}, \tag{27}$$

which is not a PRE, since it has ORed quantities that are not *disjoint*. A PRE version of it might be obtained by using the afore-mentioned disjointing procedure, namely

$$S\{0\} = X_1\{0\} \vee \bar{X}_1\{0\}\,(X_2\{0\} \vee \bar{X}_2\{0\}\,(X_3\{0\} \vee \bar{X}_3\{0\}\,X_4\{0\})). \tag{28}$$

However, a much simpler PRE is obtained by simply complementing (27), namely

$$\bar{S}\{0\} = \bar{X}_1\{0\}\;\bar{X}_2\{0\}\;\bar{X}_3\{0\}\;\bar{X}_4\{0\}. \tag{29}$$

The expression in (29) is a PRE since *ANDed* quantities in it are *statistically independent*. This example illustrates that attaining PRE form is possible not only *via* the implementation of a disjointing procedure, but also through effective utilization of statistical independence, which might be manifested in a particular form of the function and lacking in its complementary form.

5. The Boole-Shannon Expansion

The most effective way for converting a Boolean formula into a PRE form is the Boole-Shannon Expansion, which takes the following form in the two-valued case [59-61, 64, 66, 75]

$$f(\boldsymbol{X}) = (\bar{X}_i \wedge f(\boldsymbol{X}|0_{\mathrm{i}})) \vee (X_{\mathrm{i}} \wedge f(\mathbf{X}|1_{\mathrm{i}})), \quad (30)$$

This Boole-Shannon Expansion expresses a (two-valued) Boolean function $f(\boldsymbol{X})$ in terms of its two subfunctions $f(\boldsymbol{X}|0_{\mathrm{i}})$ and $f(\boldsymbol{X}|1_{\mathrm{i}})$. These subfunctions are equal to the Boolean quotients $f(\boldsymbol{X})/\bar{X}_i$ and $f(\boldsymbol{X})/X_{\mathrm{i}}$, and hence are obtained by restricting X_i in the expression of $f(\boldsymbol{X})$ to 0 and 1, respectively. If $f(\boldsymbol{X})$ is a sum-of-products (sop) expression of n variables, the two sub-functions $f(\boldsymbol{X}|0_{\mathrm{i}})$ and $f(\boldsymbol{X}|1_{\mathrm{i}})$ are functions of at most $(n-1)$ variables. A multi-valued extension of (30) is

$$S(\boldsymbol{X}) = X_i\{0\} \wedge (S(\boldsymbol{X})/X_i\{0\}) \vee X_i\{1\} \wedge (S(\boldsymbol{X})/X_i\{1\}) \vee X_i\{2\} \wedge (S(\boldsymbol{X})/X_i\{2\}) \vee X_i\{3\} \wedge (S(\boldsymbol{X})/X_i\{3\}) \vee \ldots \vee X_i\{m_i\} \wedge (S(\boldsymbol{X})/X_i\{m_i\}). \quad (31)$$

A formal proof of (31) is achieved by "perfect induction," that is, by considering the (m_i+1) exhaustive cases, namely: $\{X_i\{0\}=1\}$, $\{X_i\{1\}=1\}$, $\{X_i\{2\}=1\}$, $\{X_i\{3\}=1\}$, …, and $\{X_i\{m_i\}=1\}$. In the first case, for example, $\{X_i\{0\}=1\}$, and consequently $\{X_i\{1\}= X_i\{2\}= X_i\{3\}=\cdots= X_i\{m_i\}=0\}$. Therefore,

$$\text{The L.H.S. of (31)} = \text{the R.H.S. of (31)} = S(\boldsymbol{X})|(X_i\{0\}=1) = S(\boldsymbol{X})/X_i\{0\}. \quad (32)$$

The other m_i cases can be handled in a similar fashion. The expansion (31) serves our purposes very well. Once the sub-functions in (31) are expressed by PRE expressions, $S(\mathbf{X})$ will be also in PRE form, thanks to the combination of the following two facts:

(a) The R.H.S. of (31) has (m_i+1) disjoint terms, each of which containing one of the (m_i+1) disjoint instances $X_i\{0\}, X_i\{1\}, X_i\{2\}, X_i\{3\}$, …, and $X_i\{m_i\}$ of the variable X_i,

(b) Each of these (m_i+1) terms is a product of two statistically-independent entities, since any sub-function $S(\boldsymbol{X})/X_i\{j\}$ $(0 \leq j \leq m_i)$ does not involve any instance of the (m_i+1)-valued variable X_i, since its $X_i\{j\}$ instance is set to 1, while all its other instances are set to 0.

The expansion (31) might be viewed as a justification of the construction of the multi-valued Karnaugh map used extensively herein [30-34, 48]. This expansion transforms directly, on a one-to-one basis, to the probability domain as

$$E\{S(\boldsymbol{X})\} = E\{X_i\{0\}\} * E\{S(\boldsymbol{X})/X_i\{0\}\} + E\{X_i\{1\}\} * E\{S(\boldsymbol{X})/X_i\{1\}\} + E\{X_i\{2\}\} * E\{S(\boldsymbol{X})/X_i\{2\}\} + E\{X_i\{3\}\} * E\{S(\boldsymbol{X})/X_i\{3\}\} + \cdots + E\{X_i\{m_i\}\} * E\{S(\boldsymbol{X})/X_i\{m_i\}\}. \quad (33)$$

Equation (33) might be viewed as a restatement of the *Total Probability Theorem*, provided we interpret the expectation of a Boolean quotient as a conditional probability [59, 76, 77]. It is the basis of multi-valued decision diagrams (*MDDs)*, that are optimally employed for the reliability analysis of multi-state systems [23-25], and that constitute the multi-valued counterpart of the Binary decision diagrams [75].

6. A Non-homogeneous non Binary-Imaged Two-Variable Example

This example is taken from one of the best available textbooks on multistate reliability [15], wherein the example is solved via techniques borrowed from the theory of discrete functions [35]. The solution in [15] handles discrete functions reasonably, but then seeks probability expressions through effectively using the Inclusion-Exclusion (IE) Principle, which is notorious for its poor computational complexity and its production of prohibitively long reliability expressions that result in exaggerated round-off errors [64, 70, 78, 79]. The solution presented herein avoid the IE shortcomings through the derivation of a PRE while still in the Boolean domain. Not only is the solution procedure much simpler and more intuitive than any standard solution, such as the one in [15], but the final expressions obtained are much more compact as well. Though the current system lacks a binary image, most of its analysis herein deals solely with binary entities such as $S\{\geq j\}$, $S\{\leq j\}$, and $S\{j\}$. The ultimate goal of the analysis is to obtain $E\{S\{j\}\}$ for $j = o, 1, \ldots, M$, which might be conveniently obtained through the analysis of $S\{\geq j\}$ and $S\{\leq j\}$. It is only towards the end of this section that we deal explicitly with the multi-valued S rather than with its binary instances.

The system considered in this example is a non-homogeneous one specified by the function table of its structure or success function $S(\boldsymbol{X})$

$$S: \{0, 1, 2,3,4\} \times \{0, 1, 2, 3\} \rightarrow \{0, 1, 2, 3\}. \quad (34)$$

This function table is shown in Fig. 3, and is conveniently identified to be in the form of a multi-valued Karnaugh map (MVKM). All entries of this map are explicitly given, but this is a superfluous representation of this coherent structure function, since it suffices to specify either (a) the bold entries in the cells with blue color (the minimal upper vectors (MUVs)), or (b) the bold entries in the cells with red color (the maximal lower vectors (MLVs)), where the cell (0, 3) belongs to both sets in (a) and (b).

The following set of equations is a complete non-binary-image characterization of the system under study. They are obtained from Fig. 4(a), 5(a), and 6(a), respectively, and they give each binary function $S\{\geq j\}$ (for $j = 3, 2$ and 1), as a function of $\boldsymbol{X}$ in general (and not necessarily in terms of $\boldsymbol{X}\{\geq j\}$ (for $j = 3, 2$ and 1) alone) Here, $S\{\geq j\}$ depicts system success at level j (upper states) in a minimal sum-of-products form

$$S\{\geq 3\} = X_1\{\geq 3\}\, X_2\{\geq 3\} \quad (35a)$$

$$S\{\geq 2\} = X_1\{\geq 1\}\, X_2\{\geq 2\} \vee X_1\{\geq 3\}\, X_2\{\geq 1\} \vee X_1\{\geq 4\}\, X_2\{\geq 0\}, \quad (35b)$$

$$S\{\geq 1\} = X_1\{\geq 2\} \vee X_2\{\geq 3\} \vee X_1\{\geq 1\}\, X_2\{\geq 2\}. \quad (35c)$$

The minimal upper vectors (MUVs) at levels j $\{j = 3, 2, and\ 1\}$ can be observed (as minimal cells of upper loops) from Fig. 4(a), 5(a), and 6(a), respectively, or deduced, through immediate inspection, from equations (35) as

$$\theta(3) = \{(3, 3)\}, \quad (36a)$$

$$\theta(2) = \{(1, 2), (3,1), (4, 0)\}, \quad (36b)$$

$$\theta(1) = \{(2, 0), (0, 3), (1,2))\}. \quad (36c)$$

We reiterate that there exists a subtle difference between a minimal upper vector (MUV) at a certain level and a corresponding prime implicant of success (minimal path) at that level, despite the existence of a one-to-one relation between them. In fact, a minimal path constitutes all the upper vectors extending (inclusively) from a particular MUV to the all-highest vector. For example, Success at level 1 has three prime implicants, the first of which is

$$X_1\{\geq 2\} = X_1\{\geq 2\}\, X_2\{\geq 0\} = X_1\{2, 3, 4\}\, X_2\{0, 1, 2, 3\} = X_1\{2\}\, X_2\{0\} \vee X_1\{2\}\, X_2\{1\} \vee X_1\{2\}\, X_2\{2\} \vee X_1\{2\}\, X_2\{3\} \vee X_1\{3\}\, X_2\{0\} \vee X_1\{3\}\, X_2\{1\} \vee X_1\{3\}\, X_2\{2\} \vee X_1\{3\}\, X_2\{3\} \vee X_1\{4\}\, X_2\{0\} \vee X_1\{4\}\, X_2\{1\} \vee X_1\{4\}\, X_2\{2\} \vee X_1\{4\}\, X_2\{3\}, \quad (37)$$

comprises 12 vectors (or Karnaugh map cells, as shown in Fig. 6(a)), with its lowest vector being the MUV $X_1\{2\}\, X_2\{0\}$ (abbreviated as an ordered set $(2, 0)$ in (36c)), and with its highest vector being the all-highest vector $X_1\{4\}\, X_2\{3\}$. The fact that this system is non-binary-imaged is reflected in that the set $\theta(j)$ contains members with elements other than j and 0.

Similarly, the following set of equations is another complete non-binary-image characterization of the system under study.

They are obtained either from Fig. 4(b), 5(b), and 6(b), respectively, or by complementation and application of De Morgan's rules to the former equations (35). The new equations give each binary function $S\{\leq (j-1)\}$ (for $j = 3, 2\ and\ 1$), as a function of $\boldsymbol{X}$ in general (and not necessarily in terms of $\boldsymbol{X}\{\leq (j-1)\}$ (for $j = 3, 2\ and\ 1$) alone) Here, $S\{< j\} = S\{\leq (j-1)\}$ depicts system failure at level j (lower states), again in a minimal sum-of-products form:

$$S\{< 3\} = S\{\leq 2\} = X_1\{\leq 2\} \vee X_2\{\leq 2\}, \quad (38a)$$

$$S\{< 2\} = S\{\leq 1\} = X_1\{\leq 0\} \vee X_1\{\leq 2\}X_2\{\leq 1\} \vee X_1\{\leq 3\}\,X_2\{\leq 0\}, \quad (38b)$$

$$S\{< 1\} = S\{\leq 0\} = X_1\{\leq 0\}X_2\{\leq 2\} \vee X_1\{\leq 1\}X_2\{\leq 1\}. \quad (38c)$$

The maximal lower vectors (MLVs) at level $j\,\{j = 2, 1, and\ 0\}$ can be observed (as maximal cells of lower loops) from Fig. 4(b), 5(b), and 6(b), respectively, or deduced, through immediate inspection, from equations (38) as

$$\sigma(2) = \{(2,3), (4,2)\}. \quad (39a)$$

$$\sigma(1) = \{(0,3), (2,1), (3,0)\} \quad (39b)$$

$$\sigma(0) = \{(0,2), (1,1)\}. \quad (39c)$$

We observe that there exists also a minor difference between a maximal lower vector (MLV) at a certain level and a corresponding prime implicant of failure (minimal cutset) at that level, despite the fact that each of them uniquely specifies the other. In fact, a minimal cutset constitutes all the lower vectors extending (inclusively) from the all-0 vector to the corresponding MLV. The fact that this system is non-binary-imaged is reflected in that the set $\sigma(j)$ contains members with elements other than j and the maximal elements $m_1 = 4$ and $m_2 = 3$.

Now, we replace the success expressions by probability-ready expressions (PREs) by using the algebraic procedure in Sec. 4, or by replacing the loops in Fig. 4(a), 5(a), and 6(a), respectively, with non-overlapping loops. These PREs can be readily converted (on a one-to-one basis) into expected values by replacing the logical ORing and ANDing by arithmetic counterparts of addition and multiplication and replacing component instances by their expected values (see Table 1).

$$S_{PRE}\{\geq 3\} = X_1\{\geq 3\}\,X_2\{\geq 3\}, \quad (40a)$$

$$S_{PRE}\{\geq 2\} = X_1\{\geq 4\} \vee X_1\{< 4\}(X_1\{\geq 1\}X_2\{\geq 2\} \vee (X_1\{< 1\} \vee X_1\{\geq 1\}X_2\{< 2\})X_1\{\geq 3\}\,X_2\{\geq 1\}) = X_1\{4\} \vee X_1\{1,2,3\}\,X_2\{\geq 2\} \vee X_1\{3\}\,X_2\{1\}, \quad (40b)$$

$$S_{PRE}\{\geq 1\} = X_1\{\geq 2\} \vee X_1\{< 2\}(X_2\{\geq 3\} \vee X_2\{< 3\}X_1\{\geq 1\}X_2\{\geq 2\}) = X_1\{\geq 2\} \vee X_1\{< 2\}X_2\{\geq 3\} \vee X_1\{1\}X_2\{2\}. \quad (40c)$$

Next, we replace the failure expressions by probability-ready expressions (PREs) by using the algebraic procedure in Sec. 4, or by replacing the loops in Fig. 4(b), 5(b), and 6(b), respectively, with non-overlapping loops. Again, these PREs can be readily converted (on a one-to-one basis) into expected values by replacing the logical ORing and ANDing by arithmetic counterparts of addition and multiplication and replacing component instances by their expected values (see Table 1).

$$S_{PRE}\{\leq 2\} = X_1\{\leq 2\} \vee X_1\{> 2\}\;X_2\{\leq 2\}, \quad (41a)$$

$$S_{PRE}\{\leq 1\} = X_1\{0\} \vee X_1\{> 0\}(X_1\{\leq 2\}X_2\{\leq 1\} \vee (X_1\{> 2\} \vee X_1\{\leq 2\}X_2\{> 1\})X_1\{\leq 3\}X_2\{\leq 0\}) = X_1\{0\} \vee X_1\{1,2\}X_2\{\leq 1\} \vee X_1\{3\}X_2\{0\}, \quad (41b)$$

$$S_{PRE}\{\leq 0\} = X_1\{\leq 1\}X_2\{\leq 1\} \vee (X_1\{> 1\} \vee X_1\{\leq 1\}X_2\{> 1\})X_1\{\leq 0\}X_2\{\leq 2\} = X_1\{\leq 1\}X_2\{\leq 1\} \vee X_1\{0\}X_2\{2\}. \quad (41c)$$

The PRE expressions (40) and (41) might be directly converted (on a one-to-one basis) to their expected values by replacing the AND and OR operators with the multiplication and addition operators and replacing variable instances by their expectations, namely

$$E\{X_i\{j\}\} = p_{ij}. \quad (42)$$

$$E\{X_i\{\geq j\}\} = p_{ij} + p_{i(j+1)} + \cdots + p_{im_i} = 1 - (p_{i0} + p_{i1} + \cdots + p_{i(j-1)}). \quad (43)$$

$$E\{X_i\{\leq j\}\} = p_{i0} + p_{i1} + \cdots + p_{ij} = 1 - (p_{i(j+1)} + p_{i(j+2)} + \cdots + p_{im_i}). \quad (44)$$

We obtain expectations $E\{S\{j\}\}$ for $j = 0,1, \ldots, M$, of various instances of the multi-valued system success S, by taking differences of appropriate expectations of the forms of $E\{S\{\geq j\}\}$ and $E\{S\{\leq j\}\}$. Table 1 demonstrates the two possible alternatives for achieving this purpose. Of course, the more compact alternative is preferable.

An alternative approach is to deal with the various instances of the multi-valued system success S directly in the Boolean domain. For example, we might obtain the instance $S\{2\}$ as

$$S\{2\} = S\{\geq 2\}\, S\{\leq 2\}$$

$$= (X_1\{\geq 1\}\, X_2\{\geq 2\} \vee X_1\{\geq 3\}\, X_2\{\geq 1\} \vee X_1\{\geq 4\}\, X_2\{\geq 0\})\ (X_1\{\leq 2\} \vee X_2\{\leq 2\})$$

$$= X_1\{1,2\}\, X_2\{\geq 2\} \vee X_1\{\geq 1\}\, X_2\{2\} \vee X_1\{\geq 3\}\, X_2\{1,2\} \vee X_1\{\geq 4\}\, X_2\{0,1,2\}. \quad (45)$$

This expression can be converted to a PRE via the procedure in Sec. 4, or by covering the 2-entries in Fig. 3 by non-overlapping loops. The result is

$$S_{PRE}\{2\} = X_1\{1,2\}\, X_2\{\geq 2\} \vee X_1\{\geq 3\}\, X_2\{1,2\} \vee X_1\{4\}\, X_2\{0\}. \quad (46)$$

Equation (46) transforms to the following expectation, which is equivalent to the two corresponding results in Table 1.

$$E\{S\{2\}\} = (p_{11} + p_{12})(p_{22} + p_{23}) + (p_{13} + p_{14})(p_{21} + p_{22}) + p_{14}p_{20}. \quad (47)$$

To close this section, we note that we have so far used binary representations only to deal with the discrete multi-valued function $S(\boldsymbol{X})$. According to discrete-function theory [15, 35], this function should be expressed in a minimal sum-of-products form as

$$S = 0\, S\{\geq 0\} \vee 1\, S\{\geq 1\} \vee 2\, S\{\geq 2\} \vee 3\, S\{\geq 3\} = 1\, S\{\geq 1\} \vee 2\, S\{\geq 2\} \vee 3\, S\{\geq 3\}, \quad (48)$$

where $(A \vee B)$ denotes the maximum value of A and B, and the binary expressions $S\{\geq 1\}$, $S\{\geq 2\}$, and $S\{\geq 3\}$ are given by their minimal sum-of-products forms in (35). The overall minimality in (48) relies in the inclusion relations among these expressions

$$S\{\geq 1\} \geq S\{\geq 2\} \geq S\{\geq 3\}. \quad (49)$$

To see that Equation (48) is appropriate, let us consider, for example, the case when $S = 2$, for then $S\{\geq 1\} = S\{\geq 2\} = 1$, and $S\{\geq 3\} = 0$, so that the R.H.S. of (49) becomes $1\ (1) \vee 2\ (1) \vee 3\ (0) = 1 \vee 2 \vee 0 = \max\ (1,2,0) = 2$, as expected. Despite the convenience of minimality offered by (48), it is not adequate for producing an expectation, since it has non-disjoint terms. A simpler and more convenient expression for $S(\boldsymbol{X})$ is the pseudo-Boolean expression [57, 75, 80-82]

$$S = 1\, S\{1\} + 2\, S\{2\} + 3\, S\{3\}, \quad (50)$$

which has the corresponding expected value (thanks to the fact that expectation of an arithmetic sum is the arithmetic sum of expectations)

$$E\{S\} = 1\, E\{S\{1\}\} + 2\, E\{S\{2\}\} + 3\, E\{S\{3\}\}. \quad (51)$$

The expected value of S lies in the interval $[0, M] = [0, 3]$, and is a weighted sum of the expectations of its instances, which, in turn, are expressed as in Table 1.

X_1	0	1	2	
	0	**0**	**1**	0
	0	**1**	**2**	1
	1	**2**	2	2
				X_2

p_1	p_{10}	p_{11}	p_{12}	
	0	**0**	**1**	p_{20}
	0	**1**	**2**	p_{21}
	1	**2**	2	p_{22}
				p_2

(a) S $\xrightarrow[\text{Transform}]{\text{Probability}}$ **(b) E{S}**

Fig. 1. The structure function (success) of a small multi-state system of two three-valued components, shown in (a) with the probability transform being applied to obtain its expectation in (b). Note that the expectation function is a multi-affine function that possesses the same 'truth table' or MVKM as the logic success function. The cells with bold entries are either minimal upper vectors (MUVs) or maximal lower vectors (MLVs). The 1 entries are both MUVs and MLVs.

X_1	0	1	2	
	0	**1**	1	0
	1	1	**1**	1
	1	**1**	**2**	2
				X_2

$(a)\ \ S_a$

X_1	0	1	2	
	0	**1**	**1**	0
	1	**1**	**2**	1
	1	**2**	2	2
				X_2

$(b)\ \ S_b$

X_1	0	1	2	
	0	**0**	**2**	0
	0	**1**	2	1
	2	2	2	2
				X_2

$(c)\ \ S_c$

Fig. 2. Multi-valued Karnaugh maps (MVKMs) representing the structure functions of three small coherent three-state systems of two three-valued components, which are (a) a system that is both dominant and binary imaged, (b) a dominant system that is not binary imaged, and (c) a system that is non-dominant and hence non binary imaged.

X_1	0	1	2	3	4	
	0	0	**1**	**1**	**2**	0
	0	**0**	**1**	**2**	2	1
	0	**2**	2	2	**2**	2
	1	2	**2**	**3**	3	3
						X_2

$S(X_1, X_2)$

Fig. 3. Multi-valued Karnaugh map (MVKM) representing the structure function of the coherent multistate system of Section 6. The function is completely specified by either (a) the cells with blue bold entries (called minimal upper vectors), or (b) the cells with red bold entries (called maximal lower vectors). Note that the cell (0, 3) belongs to both sets in (a) and (b), and hence it is distinguished in violet (i.e., a mixture of blue and red).

X_1	0	1	2	3	4	
						0
						1
						2
				1	1	3
						X_2

$$(a)\ S\{\geq 3\} = S\{3\} = X_1\{\geq 3\}\, X_2\{\geq 3\}$$

X_1	0	1	2	3	4	
	1	1	1	1	1	0
	1	1	1	1	1	1
	1	1	1	1	**1**	2
	1	1	**1**			3
						X_2

$(b)\ S\{< 3\} = S\{0, 1, 2\} = S\{0\} \vee S\{1\} \vee S\{2\} == S\{\leq 2\}\ = X_1\{\leq 2\}\ \vee\ X_2\{\leq 2\}$

Fig. 4. Conventional Karnaugh maps (CKMs) for (a) success at level 3, and (b) failure at level 3 for the system of Section 6. Cells of bold entries denote the minimal upper vector at level 3: $\boldsymbol{\theta(3) = \{(3, 3)\}}$ and the maximal lower vectors at level 2: $\boldsymbol{\sigma(2) = \{(2, 3), (4, 2)\}}$.

X_1	0	1	2	3	4	
					1	0
				1	1	1
		1	1	1	1	2
		1	1	1	1	3
						X_2

$(a)\quad S\{\geq 2\} = S\{2,3\} = S\{2\} \vee S\{3\}$

$= X_1\{\geq 1\}\, X_2\{\geq 2\} \vee X_1\{\geq 3\}\, X_2\{\geq 1\} \vee X_1\{\geq 4\}\, X_2\{\geq 0\}$

X_1	0	1	2	3	4	
	1	1	1	1		0
	1	1	1			1
	1					2
	1					3
						X_2

$(b)\quad S\{< 2\} = S\{\leq 1\} = S\{0,1\} = S\{0\} \vee S\{1\}$

$= X_1\{\leq 0\} \vee X_1\{\leq 2\}\, X_2\{\leq 1\} \vee X_1\{\leq 3\}\, X_2\{\leq 0\}$

Fig. 5. Conventional Karnaugh maps (CKMs) for (a) success at level 2, and (b) failure at level 2. Cells of bold entries in (a) denote the minimal upper vectors at level 2: $\boldsymbol{\theta}(2) = \{(1,2),(3,1),(4,0)\}$, and those in (b) depict the maximal lower vectors at level 1: $\boldsymbol{\sigma}(1) = \{(0,3),(2,1),(3,0)\}$.

X_1	0	1	2	3	4	
			1	1	1	0
			1	1	1	1
		1	1	1	1	2
	1	1	1	1	1	3
						X_2

$(a)\quad S\{\geq 1\} = S\{1,2,3\} = S\{1\} \vee S\{2\} \vee S\{3\}$

$= X_1\{\geq 2\} \vee X_2\{\geq 3\} \vee X_1\{\geq 1\} X_2\{\geq 2\}$

X_1	0	1	2	3	4	
	1	1				0
	1	1				1
	1					2
						3
						X_2

$(b)\quad S\{< 1\} = S\{\leq 0\} = X_1\{\leq 0\}\, X_2\{\leq 2\} \vee X_1\{\leq 1\}\, X_2\{\leq 1\}.$

Fig. 6. Conventional Karnaugh maps (CKMs) for (a) success at level 1, and (b) failure at level 1. Cells of bold entries denote the minimal upper vectors at level 1: $\boldsymbol{\theta}(1) = \{(2,0),(0,3),(1,2)\}$ and the maximal lower vectors at level 0: $\boldsymbol{\sigma}(0) = \{(0,2),(1,1)\}$. The vectors $(1,2)$ is a minimal upper vector for both levels 1 and 2.

Table 1. Expectation of Success instances of the system in example 1 via two different methods.

Expectation of an instance	Calculated via minimal upper vectors	Calculated via maximal lower vectors
$E\{S\{3\}\}$	$E\{S\{\geq 3\}\} = (p_{13} + p_{14})p_{23}$	$E\{S\{\leq 3\}\} - E\{S\{\leq 2\}\} = 1 - (p_{10} + p_{11} + p_{12} + (p_{13} + p_{14})(p_{20} + p_{21} + p_{22})$
$E\{S\{2\}\}$	$E\{S\{\geq 2\}\} - E\{S\{\geq 3\}\} = p_{14} + (p_{11} + p_{12} + p_{13})(p_{22} + p_{23}) + p_{13}p_{21} - (p_{13} + p_{14})p_{23} = p_{14} + (p_{11} + p_{12} + p_{13})p_{22} + (p_{11} + p_{12} - p_{14})p_{23} + p_{13}p_{21}$	$E\{S\{\leq 2\}\} - E\{S\{\leq 1\}\} = (p_{10} + p_{11} + p_{12}) + (p_{13} + p_{14})(p_{20} + p_{21} + p_{22}) - (p_{10} + (p_{11} + p_{12})(p_{20} + p_{21}) + p_{13}p_{20})$
$E\{S\{1\}\}$	$E\{S\{\geq 1\}\} - E\{S\{\geq 2\}\} = (p_{12} + p_{13} + p_{14}) + (p_{10} + p_{11})p_{23} + p_{11}p_{22} - (p_{14} + (p_{11} + p_{12} + p_{13})(p_{22} + p_{23}) + p_{13}p_{21})$	$E\{S\{\leq 1\}\} - E\{S\{\leq 0\}\} = p_{10} + (p_{11} + p_{12})(p_{20} + p_{21}) + p_{13}p_{20} - \left((p_{10} + p_{11})(p_{20} + p_{21}) + p_{10}p_{22}\right) = (p_{12} + p_{13})p_{20} + p_{12}p_{21} + p_{10}p23$
$E\{S\{0\}\}$	$E\{S\{\geq 0\}\} - E\{S\{\geq 1\}\} = 1 - ((p_{12} + p_{13} + p_{14}) + (p_{10} + p_{11})p_{23} + p_{11}p_{22}$	$E\{S\{\leq 0\}\} = (p_{10} + p_{11})(p_{20} + p_{21}) + p_{10}p_{22}$

7. A Homogenous Binary-Imaged Three-Variable Example

In this example, we revisit example 3 of Wood [8]. A four-valued coherent system consists of three four-valued components. The system has the following verbal description, which implies that the system has a binary image

1. A series system (A 3-out-of-3: G system, i.e., a 1-out-of-3: F system) at level 3.
2. A 2-out-of-3: G system (A 2-out-of-3: F system) at level 2.
3. A parallel system (A 1-out-of-3: G system, i.e., a 3-out-of-3: F system) at level 1.

This verbal description translates to the algebraic description of system successes at the respective levels, which constitutes an up binary image of the system

$$S\{\geq 3\} = X_1\{\geq 3\}X_2\{\geq 3\}X_3\{\geq 3\}, \tag{52a}$$

$$S\{\geq 2\} = X_1\{\geq 2\}X_2\{\geq 2\} \vee X_1\{\geq 2\}X_3\{\geq 2\} \vee X_2\{\geq 2\}X_3\{\geq 2\}, \tag{52b}$$

$$S\{\geq 1\} = X_1\{\geq 1\} \vee X_2\{\geq 1\} \vee X_3\{\geq 1\}. \tag{52c}$$

The verbal description can also be converted to the following algebraic description of system failure at the respective levels, which constitutes a down binary image of the system

$$S\{<3\} = X_1\{<3\} \vee X_2\{<3\} \vee X_3\{<3\}, \quad (53a)$$

$$S\{<2\} = X_1\{<2\}X_2\{<2\} \vee X_1\{<2\}X_3\{<2\} \vee X_2\{<2\}X_3\{<2\}, \quad (53b)$$

$$S\{<1\} = X_1\{<1\}X_2\{<2\}X_3\{<1\}. \quad (53c)$$

Equations (53) can be also obtained through complementation of equations (52). Equations (52) can be rewritten in terms of the minimal upper vectors (MUVs)

$$\theta(3) = \{(3,3,3)\}, \quad (54a)$$

$$\theta(2) = \{(2,2,0),(2,0,2),(0,2,2)\}, \quad (54b)$$

$$\theta(1) = \{(1,0,0),(0,1,0),(0,0,1)\}. \quad (54c)$$

Note that a prime implicant of the form $X_1\{\geq 2\}X_2\{\geq 2\}$ (in which the variable X_3 is absent) is equivalent to $X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 0\}$ and hence it lead to the MUV $(2,2,0)$. For this binary-imaged system, elements of $\theta(j)$ are vectors of j or 0 components only.

Equations (53) might be rewritten with the symbols $\{<3\}, \{<2\}$ and $\{<1\}$ replaced by $\{\leq 2\}, \{\leq 1\}$ and $\{\leq 0\}$, namely

$$S\{\leq 2\} = X_1\{\leq 2\} \vee X_2\{\leq 2\} \vee X_3\{\leq 2\}, \quad (55a)$$

$$S\{\leq 1\} = X_1\{\leq 1\}X_2\{\leq 1\} \vee X_1\{\leq 1\}X_3\{\leq 1\} \vee X_2\{\leq 1\}X_3\{\leq 1\}, \quad (55b)$$

$$S\{\leq 0\} = X_1\{\leq 0\}X_2\{\leq 0\}\, X_3\{\leq 0\}. \quad (55a)$$

Now, equations (55) (with the $\leq$ notation) tells us that the maximal lower vectors (MLVs) are

$$\sigma(2) = \{(2,3\,,3),(3,2,3),(3,3,2)\}, \quad (56a)$$

$$\sigma(1) = \{(1,1,3),(1,3,1),(3,1,1)\}, \quad (56b)$$

$$\sigma(0) = \{(0,0,0)\}. \quad (56c)$$

By contrast to the case of the MUVs in which an absent variable X_i stands for $X_i\{\geq 0\}$ and is expressed by 0 in the MUV, the present case of MLUs has an absent variable X_i standing for $X_i\{\leq 3\}$ and being expressed by 3 in the MLU. For this binary-imaged system, elements of $\sigma(j)$ are vectors of j or 3 components only.

Figure 7 displays a multi-valued Karnaugh map (MVKM) representing the structure function of the present coherent multistate system. The function is completely specified by either (a) the cells with bold blue entries, which are the minimal upper vectors of (54), or (b) the cells with bold red entries, which are the maximal lower vectors of (56). Figure 8 displays three conventional Karnaugh maps (CKMs) for the binary successes at levels 3, 2, and 1, while Fig. 9 shows CKMs for the binary failures at levels 1, 2, and 3. Equations (52) can be converted to PREs via the procedure in Sec. 4

$$S_{PRE}\{\geq 3\} = X_1\{\geq 3\}X_2\{\geq 3\}X_3\{\geq 3\}, \quad (57a)$$

$$S_{PRE}\{\geq 2\} = X_1\{\geq 2\}\, X_2\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{<2\}\, X_3\{\geq 2\} \vee X_1\{<2\}\, X_2\{\geq 2\}X_3\{\geq 2\}, \quad (57b)$$

$$S_{PRE}\{\geq 1\} = X_1\{\geq 1\} \vee X_1\{<1\}\, (X_2\{\geq 1\} \vee X_2\{<1\}\, X_3\{\geq 1\}). \quad (57c)$$

Likewise, equations (55) can be converted to PREs via the procedure in Sec. 4

$$S_{PRE}\{\leq 2\} = X_1\{\leq 2\} \vee\ X_1\{>2\}(X_2\{\leq 2\} \vee X_2\{>2\}\, X_3\{\leq 2\}), \quad (58a)$$

$$S_{PRE}\{\leq 1\} = X_1\{\leq 1\}X_2\{\leq 1\} \vee X_1\{\leq 1\}\, X_2\{>1\}\, X_3\{\leq 1\} \vee\ X_1\{>1\}\, X_2\{\leq 1\}X_3\{\leq 1\}, \quad (58b)$$

$$S_{PRE}\{\leq 0\} = X_1\{\leq 0\}X_2\{\leq 0\}\, X_3\{\leq 0\}. \quad (58c)$$

Equations (57b) and (58b) are demonstrated by the blue and red non-overlapping loops in Fig. 10, respectively.

8. The Issue of Duality

Reliability analysis inherits the meta-mathematical principle of duality from Boolean algebra. We call two systems dual if they have dual structure functions. The dual of a multi-valued function $S(\boldsymbol{X})$ is labelled $S^d(\boldsymbol{X})$ and given by

$$S^d(X_1, X_2, \dots, X_n) = \bar{S}(\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n), \qquad (59)$$

where complementation of the output variable S to $\bar{S}$ means the replacement of every value j of it by $(M - j)$, and complementation of an input variable X_k to $\bar{X}_k$ means the replacement of every value j of it by $(m_k - j)$. Therefore, the dual function S^d can be written as [9, 82-86]

$$S^d(\boldsymbol{X}) = M - S(\boldsymbol{m} - \boldsymbol{X}). \qquad (60)$$

where $\boldsymbol{m} - \boldsymbol{X} = ((m_1 - X_1), (m_2 - X_2), \dots, (m_k - X_k), \dots, (m_n - X_n))$. Since the $S(\boldsymbol{X})$ function can be given by a generalization of (48) as

$$S(\boldsymbol{X}) = \vee_{j=1}^{M} \; j \;\; S\{\geq j\}(\boldsymbol{X}), \qquad (61)$$

Then, its dual $S^d(\boldsymbol{X})$ is given by

$$S^d(\boldsymbol{X}) = \vee_{j=1}^{M} \; (M - j) \;\; S\{\geq (M - j)\}(\boldsymbol{m} - \boldsymbol{X}). \qquad (62)$$

The multi-valued Karnaugh map (MVKM) used herein offers a handy means for obtaining the dual of the structure function of a multi-state system. Both map indices (inputs) and entries (output) of the MVKM of the dual function are obtained via complementation of those of the MVKM of the original function. Figures 11 and 12 offer MVKM representations of the systems dual to the ones discussed in Sections 6 and 7, respectively. The figures verify the observation made earlier in [9, 84] that a vector $\boldsymbol{X}$ is an MUV (MLV) of a certain level j for the original function if and only if the complementary vector $(\boldsymbol{m} - \boldsymbol{X})$ is an MLV (MUV) of the complementary level $(M - j)$ for the dual function. A comparison of the MVKMs in Fig. 7 and 12 reveals that they represent the same structure function. This means that the system in Sec. 7 (with the structure function of Fig. 7) is a self-dual one. This fact should have been anticipated, since this system has three levels with (a) dual systems (series and parallel) at the complementary levels 1 and 3, and (b) a self-dual system at the intermediate level 2.

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
	0	1	1	1	1	1	1	1	1	1	2	2	1	1	2	2	0
	1	1	1	1	1	1	1	1	1	1	2	2	1	1	2	2	1
	1	1	2	2	1	1	2	2	2	2	2	2	2	2	2	2	2
	1	1	2	2	1	1	2	2	2	2	2	2	2	2	2	3	3
																	X_3

$S(X_1, X_2, X_3)$

Fig. 7. A multi-valued Karnaugh map (MVKM) representing the structure function of the coherent multistate system of Section 7. The function is completely specified by either (a) the cells with blue bold entries (the minimal upper vectors), or (b) the cells with red bold entries (the maximal lower vectors).

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
																	0
																	1
																	2
																1	3
																	X_3

(a) $S\{\geq 3\} = S\{3\}$

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
											1	1			1	1	0
											1	1			1	1	1
			1	1			1	1	1	1	1	1	1	1	1	1	2
			1	1			1	1	1	1	1	1	1	1	1	1	3
																	X_3

(b) $S\{\geq 2\} = S\{2, 3\}$

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	3
																	X_3

$S\{\geq 1\} = S\{1, 2, 3\}$

Fig. 8. Conventional Karnaugh maps (CKMs) for the binary successes at levels 3, 2, and 1 for the system of Section 7.

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
	1																0
																	1
																	2
																	3
																	X_3

$S\{< 1\} = S\{\leq 0\} = S\{0\}$

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
	1	1	1	1	1	1	1	1	1	1			1	1			0
	1	1	1	1	1	1	1	1	1	1			1	1			1
	1	1			1	1											2
	1	1			1	1											3
																	X_3

$S\{< 2\} = S\{\leq 1\} = S\{0, 1\}$

X_1	0				1				2				3				
X_2	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		3
																	X_3

$S\{< 3\} = S\{\leq 2\} = S\{0, 1, 2\}$

Fig. 9. Conventional Karnaugh maps (CKMs) for the binary failures at levels 1, 2, and 3 for the system of Section 7.

X_2	0				1				2				3				
X_3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	
											1	1			1	1	0
											1	1			1	1	1
			1	1			1	1	1	1	1	1	1	1	1	1	2
			1	1			1	1	1	1	1	1	1	1	1	1	3
																	X_1

$$S_{PRE}\{\geq 2\} = X_1\{\geq 2\}\, X_2\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{< 2\}\, X_3\{\geq 2\} \vee X_1\{< 2\}\, X_2\{\geq 2\} X_3\{\geq 2\}$$

$$S_{PRE}\{\leq 1\} = X_1\{\leq 1\} X_2\{\leq 1\} \vee X_1\{\leq 1\}\, X_2\{> 1\}\, X_3\{\leq 1\} \vee X_1\{> 1\}\, X_2\{\leq 1\} X_3\{\leq 1\}$$

Fig. 10. A conventional Karnaugh maps (CKMs) with non-overlapping loops comprising PRE representations for the binary success (blue) and binary failure (red) at level 2 for the system of Section 7.

X_1	4	3	2	1	0	
	3	3	2	2	1	3
	3	3	2	1	1	2
	3	1	1	1	1	1
	2	1	1	0	0	0
						X_2

$$S^d(X_1, X_2) = \bar{S}(\bar{X}_1, \bar{X}_2)$$

Fig. 11. A Multi-value Karnaugh map representing the structure function of the system that is dual to the one in Section 6 (Fig. 3). Both map indices (input) and entries (output) are obtained via complementation of those in Fig. 3. Again the function is completely specified by either its minimal upper vectors (blue) or maximal lower vectors (red). Note that, for this particular map arrangement, the MUVs and the MLVs of a certain level for the original function are replaced by MLVs and MUVs of the complementary level for the dual function.

X_1	3				2				1				0				
X_2	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1	0	
	3	2	2	2	2	2	2	2	2	2	1	1	2	2	1	1	3
	2	2	2	2	2	2	2	2	2	2	1	1	2	2	1	1	2
	2	2	1	1	2	2	1	1	1	1	1	1	1	1	1	1	1
	2	2	1	1	2	2	1	1	1	1	1	1	1	1	1	0	0
																	X_3

$$S^d(X_1, X_2, X_3) = \bar{S}(\bar{X}_1, \bar{X}_2, \bar{X}_3)$$

Fig. 12. A Multi-value Karnaugh map representing the structure function of the system that is dual to the one in Section 7 (Fig. 7). Both map indices (input) and entries (output) are obtained via complementation of those in Fig. 7. Again the function is completely specified by either its minimal upper vectors (blue) or maximal lower vectors (red). Note that, for this particular map arrangement, the MUVs and the MLVs of a certain level for the original function are replaced by MLVs and MUVs of the complementary level for the dual function. Comparison of Fig. 7 and 12 asserts that this multi-state system is self-dual.

9. Conclusions

This paper utilizes algebraic and map tools for the reliability characterization and analysis of general multi-state coherent systems, which are interpreted herein to be non-repairable systems with independent non-identical components. The paper presents switching-algebraic expressions of both system success and system failure at each non-zero level. These expressions are given as minimal sum-of-products formulas or as probability–ready expressions. The paper also utilizes a convenient map representation via the multi-valued Karnaugh map for the system structure function S, or via M maps of binary entries and multi-valued inputs representing the success/failure at every non-zero level of the system. Further system characterizations are also given in terms of minimal upper vectors or maximal lower vectors. Great emphasis is placed on making a minimal departure from binary concepts and techniques, while taking care to clarify novel issues that emerged due to generalizations introduced by the multi-state model.

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تحليل رمزي مستند للجبر البولاني لمعولية النظم المتسقة متعددة الحالات ذات المركبات غير المتجانسة

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المستخلص. تختص ورقة البحث هذه بالتحليل الرمزي المستند إلى الجبر البولاني للنظم المتسقة متعددة الحالات غير القابلة للإصلاح التي مركباتها مستقلة غير متطابقة ومتعددة الحالات. نقوم بتكييف العديد من المفاهيم والأدوات الثنائية مثل التعبير الجاهز للتحول لاحتمال، وخارج القسمة البولانية، ومفكوك بول وشانون، وخريطة كارنوه وذلك عند تعدد الحالات. تستخدم الورقة الأساليب الجبرية للمنطق متعدد القيم لتقييم المستويات المتعددة لخرج النظام كدوال ثنائية أو إخبلرية لمدخلات النظام متعددة القيم. يتم بعد ذلك كتابة صيغة النجاح لكل من هذه المستويات كتعبير جاهز للتحول لاحتمال، مما يسمح بتحويلها الفوري، على أساس واحد لواحد، إلى احتمال أو قيمة متوقعة. أكملنا تحليل المعولية الرمزي لنظامين صغيرين (يمكن اعتبارهما نظامين نمطيين معياريين للنظم المتسقة متعددة الحالات) في هذه الورقة بنجاح، مما أسفر عن نتائج تم التحقق منها بشكل رمزي، كما تبين أنها تتوافق عدديًا مع تلك التي تم الحصول عليها سابقًا. أردفنا الطرائق الجبرية المستخدمة باستخدام التصور التوضيحي بواسطة خرائط كارنوه متعددة القيم. يتسم العمل هنا بالتأكيد والتركيز على تعميم مفاهيم النظم الثنائية المتسقة إلى تلك الخاصة بالنظم المتسقة متعددة الحالات، بدلا من ابتداع مفاهيم جديدة غير مألوفة قائمة بذاتها لهذه النظم الأخيرة.

الكلمات المفتاحية: معولية النظم، التعبير الجاهز للتحول لاحتمال، النظام متعدد الحالات، المنطق متعدد القيم، المتجه الأعلى الأصغري، المتجه الأدنى الأعظمي.