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Selective Deduction with the Aid of the Variable-Entered Karnaugh Maps

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ABSTRACT. An important class of logical reasoning problems involves selective deduction from given hypotheses, which is deduction with the knowledge of certain information, or the lack thereof, about some of the pertinent variables. This note solves selective deduction problems via a powerful manual pictorial tool, viz., the variable-entered Karnaugh map (VEKM). The VEKM can be utilized to obtain the set of all prime implicants of the underlying function and then apply the knowledge available to select a subset thereof. Moreover, it allows a skipping of the tedious step of obtaining the complete sum in its general form by incorporating the available knowledge right from the outset. This is a time-saving short cut indeed, since no implicants are generated except those that are ultimately retained. An illustrative example serves to explain the steps of the method proposed herein and to demonstrate its superiority to the conventional method.

1. Introduction

The modern technique of Boolean or syllogistic reasoning as formulated by Blake [1] and expounded by Brown [2] makes use of just one rule of inference, rather than the many rules conventionally employed in classical texts of symbolic logic [3]. Syllogistic reasoning is applicable in general to any Boolean algebra and in particular to the algebra of propositions or the algebra of classes. Its strategy is to chain forward from the premises, represented by an equation of the form $f = 0$, until the complete sum of f is obtained, which means that all prime implicants of f (or equivalently all prime consequents of $f = 0$) are generated. There is a striking duality between syllogistic reasoning and resolution-based techniques employed in predicate logic [4].

An important class of Boolean reasoning problems involves selective deduction, i.e., deduction restricted by certain information, or the lack thereof, about some of the pertinent variables. Selective deduction problems can be solved essentially by the technique of syllogistic reasoning; the complete sum of the pertinent function is obtained and then the knowledge available is incorporated into it through appropriate elimination processes.

This note solves selective deduction problems via a powerful manual tool, viz., the variable-entered Karnaugh map (VEKM), which is the natural map for handling a general Boolean function, defined on a Boolean algebra B that is not necessarily two valued [5,6]. The VEKM can be used for implementing the technique of syllogistic reasoning through the derivation of the pertinent complete sum [7,8]. However, the VEKM can be utilized more efficiently, since it allows incorporating the available knowledge right from the outset, which leads to a skipping of the tedious step of obtaining the underlying complete sum in its most general form. This is a time-saving short cut indeed, since only the necessary implicants are generated.

2. The Conventional Method of Selective Deduction

The conventional method of selective deduction proceeds by generating the complete sum *CS(f)* of the underlying Boolean function *f*, and then retaining only those prime implicants that are consistent with the available knowledge. The method is best understood by way of the following example which is obtained from Brown [2].

Example 1

In enzyme biochemistry, it is neither easy to isolate an enzyme in pure form, nor to observe separate chemical reactions directly. A chemist is studying enzymes A, B, and C in relation to reactions X, Y, and Z. He has made the following five observations:

1. A solution having neither A, nor B, nor C had reaction Y but neither X nor Z.
2. When the solution contained A and either B or C or both, the reaction was neither Y nor was it X and Z together.
3. When the solution had B but not A, or did not have B but had C, reactions X and Y occurred, or reaction X did not occur but Z did.
4. When the solution contained C, together with A or B or both, or else had neither A nor C, either reaction X did not take place, or both Y and Z did.
5. A solution containing A but not B either failed to produce reaction X or failed to produce reaction Z.

Based upon these observations, answers are sought, in the simplest possible form, to the following questions:

(i) What is known concerning the reactions X, Y, and Z, independent of any knowledge of enzyme content?

(ii) What is known about A, B, and C, given each of the following reactions? (i) X occurred; (ii) X did not occur; (iii) Z occurred; (iv) Z did not occur.

The conventional approach [2] expresses the information provided by these observations by a system of five conditionals and then reduces it to a single equation $f = 0$, where f is expressed by

$$\begin{aligned} f \;=\; & \bar{A}\,\bar{B}\,\bar{C}\,(X \vee \bar{Y} \vee Z) \vee A(B \vee C)\,(Y \vee XZ) \\ & \vee\, (\bar{A}B \vee \bar{B}C)\,(X\bar{Y} \vee \bar{X}\,\bar{Z}) \\ & \vee\, (AC \vee BC \vee \bar{A}\,\bar{C})\,X\,(\bar{Y} \vee \bar{Z}) \vee A\,\bar{B}\,XZ \end{aligned} \qquad (1)$$

Now, the complete sum of f is obtained as

$$\begin{aligned} CS\,(f) \;=\; & ACX \vee AXZ \vee ACY \vee A\,\bar{B}\,C\,\bar{Z} \vee ABY \vee \bar{A}\,X\,\bar{Y} \\ & \vee\, CX\bar{Y} \vee X\bar{Y}Z \vee \bar{A}\,\bar{Y}\,\bar{Z} \vee \bar{B}\,C\,\bar{Y}\,\bar{Z} \vee \bar{A}\,C\,\bar{X}\,\bar{Z} \\ & \vee\, C\,\bar{X}\,Y\,\bar{Z} \vee \bar{B}\,C\,\bar{X}\,\bar{Z} \vee \bar{A}\,\bar{B}\,\bar{C}\,X \vee \bar{B}\,\bar{C}\,XZ \\ & \vee\, \bar{A}\,\bar{B}\,\bar{C}\,\bar{Y} \vee \bar{A}\,\bar{B}\,\bar{C}\,Z \vee \bar{A}\,B\,\bar{Z} \vee BCX\bar{Z} \vee BY\bar{Z} \\ & \vee\, \bar{A}\,\bar{C}\,X\,\bar{Z}. \end{aligned} \qquad (2)$$

The step of going from (1) to (2) is very tedious. We achieved it via our improved Tison method [7] using 6 tabular constructs in which 106 consensi are generated and 102 terms are absorbed though only 21 terms are ultimately retained.

To answer question (a) eliminate the variables A, B and C (which are not of current interest) from *CS(f)* by simply deleting every term involving A, B, or C in *CS(f)*. The result, $X\,\bar{Y}\,Z = 0$, takes the clausal form, $XZ \rightarrow Y$, and means that, independent of any knowledge of enzyme content: if reactions X and Z occur together, then reaction Y occurs also. To answer parts (i) and (ii) of question (b), eliminate Y and Z from *CS (f)* = 0; to answer parts (iii) and (iv) of (b), eliminate X and Y. The resultants of elimination are:

Eliminating Y and Z: $\bar{A}\,\bar{B}\,\bar{C}\,X \vee ACX = 0,$

Eliminating X and Y: $A\ \bar{B}\ C\ \bar{Z} \quad \vee \quad \bar{A}\ \bar{B}\ \bar{C}\ Z \quad \vee \quad \bar{A}\ B\ \bar{Z} \quad = 0,$

which leads to answers for various parts of question (b), in clausal form, as follows:

(i) $A\,C \;\rightarrow\; 0; \qquad 1 \;\rightarrow\; A \vee B \vee C,$

(ii) No information,

(iii) $1 \;\rightarrow\; A \vee B \vee C,$

(iv) $A\,C \;\rightarrow\; B; \qquad B \;\rightarrow\; A.$

3. A VEKM Procedure for Selective Deduction

In problems requiring selective deduction, it is desirable to skip the tedious task of obtaining the complete sum explicitly. All we need to obtain is the set of prime implicants of the Boolean function that are independent of certain variables. This set is known to comprise the complete sum of the conjunctive eliminant of the original function with respect to the set of variables that we lack any information about [2, 8-10]. Here, a utilization of the VEKM allows us to take advantage of the interchangeability of the operation of complete sum generation (which leads to more complex expressions) with that of conjunctive elimination with respect to a set of variables (which leads to simpler expressions), thereby keeping the overall complexity within acceptable limits throughout the solution. Details of the proposed VEKM method are explained by the example to follow.

Example 2 (Example 1 revisited)

The six-variable Boolean function f in (1) is now expanded [5] about its variables A, B and C and subsequently represented by a VEKM of map variables A, B and C in Fig. 1, which is later modified in Fig.2 to have complete-sum entries. We further introduce the symbol *CE(f, A)* to stand for the conjunctive eliminant of f with respect to the set of arguments A [2] which equals that part of *CS(f)* involving variables that do not belong to the set A. It is produced by ANDing of the $2^{|A|}$ subfunctions of f obtained by restricting its value through all possible assignments of the variables belonging to A, where |A| means the cardinality or number of elements in set A. Now, we answer the questions posed in this problem as follows:

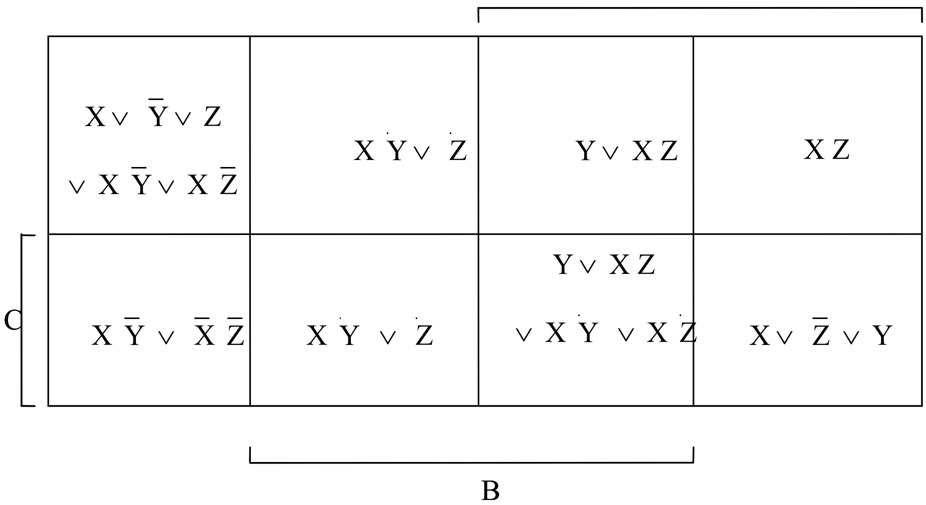

Fig. 1. A VEKM representation of the function f in (1).

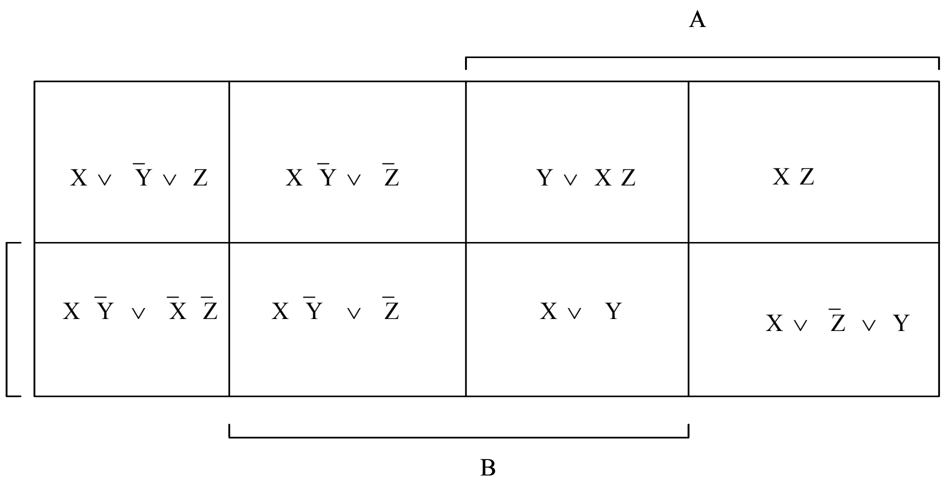

Fig. 2. The VEKM in Fig. 1 re-drawn to have complete sum entries.

(i) What is known concerning the reactions X, Y and Z, independent of any knowledge of enzyme content A, B and C is

$$0 = CE(CS(f), \{A, B, C\}) = CS(CE(f, \{A, B, C\})$$

$$= CS(\text{ANDing all VEKM cells}) = CS(X\ \bar{Y}\ Z) = X\ \bar{Y}\ Z.$$

(ii) What is known concerning the enzymes A, B, C lacking any information on Y is

$$0 = CE(CS(f), \{Y\}) = CS(CE(f, \{Y\})).$$

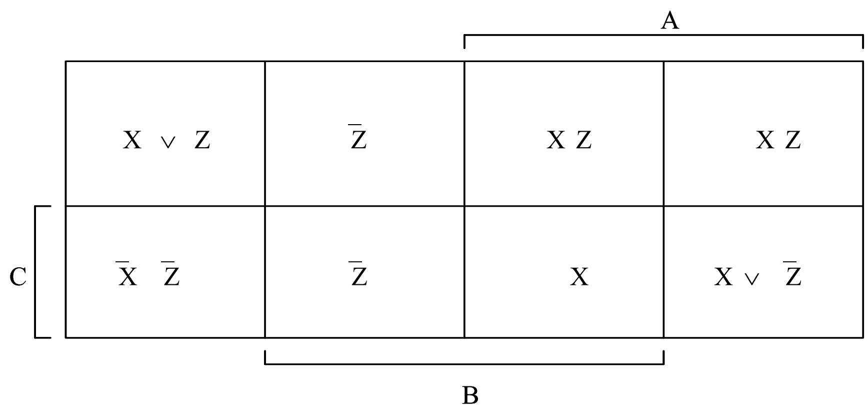

Fig. 3. A VEKM representation of CE (f, {Y}) for f given in Fig. 2.

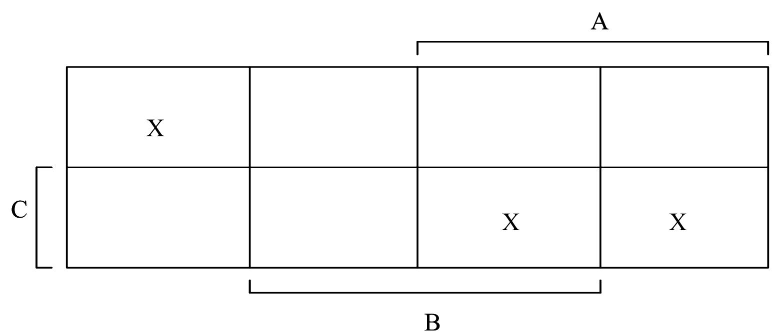

Fig. 4. A VEKM representation of CE (f, {Y, Z }) for f given in Fig. 2.

Now, construct a VEKM representation of *CE(f, {Y})* (Fig. 3) from the one of *f* with complete-sum entries (Fig. 2) by retaining in each cell in Fig. 2 only those prime implicants that are independent of Y. For questions (i) and (ii) further eliminate the variable Z to obtain *CE(f,{Y, Z })* as shown in Fig. 4. Using any method for complete-sum generation [7.8], we obtain

$$0 = CS\,(CE\,(f, \{Y, Z\}) = X\,(\bar{A}\,\bar{B}\,\bar{C} \vee AC),$$

and the answers to parts (i) and (ii) become

$$0 = CS\,(\,CE\,(f, \{\,Y, Z\,\}))|_{X=1} = \bar{A}\,\bar{B}\,\bar{C} \vee AC\,,$$

$$0 = CS\,(\,CE\,(f, \{\,Y, Z\,\}))|_{X=0} = 0.$$

Now, to answer parts (iii) and (iv) further eliminate the variable X from *CE({f,{Y})* in Fig. 3 to obtain *CE({f,{X,Y})* in Fig. 5. Again, using any method for complete-sum generation [7,6], we obtain

$$0 = CS\,(\,CE\,(f, \{X, Y\})) = Z\,(\bar{A}\,\bar{B}\,\bar{C}) \vee \bar{Z}\,(A\,\bar{B}\,C \vee \bar{A}\,B),$$

and the answers to parts (iii) and (iv) become

$$0 = CS\,(\,CE\,(f, \{\,X, Y\,\}))|_{\,Z=1} = \bar{A}\,\bar{B}\,\bar{C},$$

$$0 = CS\,(\,CE\,(f, \{\,X, Y\,\}))|_{\,Z=0} = A\,\bar{B}\,C \vee \bar{A}B.$$

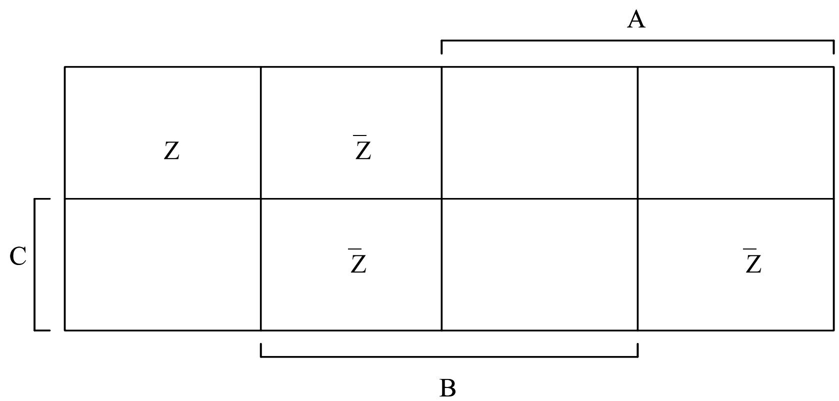

Fig. 5. A VEKM representation of CE (f, { X, Y }) for f given in Fig. 2.

The present procedure is much simpler than the earlier one in Example 1. We note, in particular, that the algebraic complexity has been kept to a minimum throughout the present solution. The process of complete-sum generation is distributed to several jobs, each of which requires at most a single iteration of the iterative consensus method [2]. Had any of these jobs been more complex, it could have been handled still readily via the VEKM itself [7,8].

4. Conclusion

A powerful VEKM procedure has been introduced for the efficient manual solution of medium-size selective deduction problems. The procedure incorporates available knowledge right from the beginning thereby avoiding the generation of any unnecessary implicants. A classical example illustrates the procedure and also serves to demonstrate its superiority to the conventional method.

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الاستنباط الاختياري بمساعدة خريطة كارنوه متغيرة المحتويات

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المستخلص: إن أحد الأصناف الهامة لمسائل الاستدلال المنطقي يتعلق بالاستنباط أو الاستنتاج الاختياري من مجموعة من الافتراضات المعطاة، وهو الاستنباط المحكوم بمعرفة أو جهل معلومات معينة حول بعض المتغيرات المعنية. تقوم هذه المقالة المقتضبة بحل مسائل متوسطة الحجم للاستنباط الاختياري باستخدام أداة يدوية تصويرية قوية هي خريطة كارنوه متغيرة المحتويات (خ ك غ ح). يمكن استغلال هذه الخريطة للحصول على المجموعة المكونة من جميع الضامنات الأولية للدالة المعنية ومن ثم إعمال المعلومات المتاحة لاختيار مجموعة جزئية من هذه المجموعة. وفضلاً عن ذلك، تسمح هذه الخريطة بتجاوز الخطوة المضنية المطلوبة للحصول على المجموع الكامل في صورته العامة وذلك بإعمال المعلومات المتاحة من بداية العمل مباشرة. ويمثل ذلك طريقًا مختصرًا موفرًا للوقت حيث لا يتم توليد أية ضامنات إلا التي تلزم في النهاية. يتم تقديم مثال تفصيلي لتوضيح خطوات الطريقة المقترحة هنا ولبيان تفوقها على الطريقة التقليدية.