On the Diophantine Equation $x^2 = 4q^n - 4q^m + 9$

Fadwa S. Abu Muriefah and Amal Al-Rashed

Riyadh University for Girls, Riyadh, Saudi Arabia
abumuriefah@yahoo.com

Abstract. In this paper, we study the title equation with $q$ any prime and $n > m \geq 0$, and we give a complete solution when $m > 0$.

Keywords: Diophantine equation, primitive divisor, Lehmer pair.

Introduction

In 1913 the Indian mathematician S. Ramanujan\cite{Ramanujan1} conjectured that the equation $x^2 = 2^n - 7$, had the only following solutions:

\begin{align*}
n & = 3 \quad 4 \quad 5 \quad 7 \quad 15 \\
x & = 1 \quad 3 \quad 5 \quad 11 \quad 181
\end{align*}

This conjecture was first proved by Nagell\cite{Nagell2} in 1948. There followed during the period 1950-1970, a number of proofs based on a variety of methods (see for example\cite{Gupta3, Gupta4}).

Ramanujan equation has the general form

$$x^2 = 4q^n - 4q^m + D,$$

where $D$ is any odd integer. The purpose of this paper is to solve

$$x^2 = 4q^n - 4q^m + 9 \quad (1)$$

with $x > 0$, $n \geq m \geq 0$, where $q$ is any prime, it is clear that $x$ is an odd integer. To solve equation (1) we will use unique factorization of ideals along with linear recurrences and congruences and the BHV Theorem\cite{BHV}. We start with the case $n = 1$ and $m = 0$. 

135
Case I

Let $\left(n, m\right) = (1,0)$, in equation (1), then we have

\[ q = \frac{x^2 - 5}{4}. \]

Since $x$ is odd, let $x = 2c + 1$, then we get

\[ q = c^2 + c - 1, \quad (2) \]

where $c$ is a positive integer. It is not known if equation (2) has infinitely many solutions.

Case II

Let $d = \gcd(m,n)$, $q_1 = q^d$, $n_1 = n/d$, $m_1 = m/d$, in equation (1), then we get the same kind of equation (1)

\[ x^2 = 4q_1^{n_1} - 4q_1^{m_1} + 9, \]

with $n_1, m_1$ are coprime. So we shall suppose $(n, m) = 1$, which means that $n \neq m$.

Case III

If $m = 0$, and $n$ is an even, then equation (1) has no solutions. So we shall exclude all the above cases.

Now we suppose the case $m > 0$, and get the following:

Theorem

The diophantine equation

\[ x^2 = 4q^n - 4q^m + 9, \quad n > m, \quad (3) \]

has the following two cases:

i. $m = 1$: When $q = 2$, then it has a unique solution given by $(x,n) = (3,1)$, otherwise it has at most two solutions
On the Diophantine Equation $x^2 = 4q^n - 4q^m + 9$

ii. $m > 1$: (a) When $m$ is odd, it has solutions only if $q = 3$, and these solutions are given by

$$(x,n,m) = (93,7,3), (2.3^{m-1} - 3, 2(m-1), m).$$

(b) When $m$ is even, it has solutions only if $q = 2$ and $m = 2$, and these solutions are given by

$$(x,n) = (1,1), (3,2), (5,3), (11,5), (181,13).$$

**Proof**

(i) Let $m = 1$ in (3).

If $q = 2$, then we get equation $x^2 - 1 = 2^{n+2}$, which is clear has a unique solution $(x,q) = (3,2)$. If $q \neq 2$, then the equation $x^2 = 4q^n - 4q^m + 9$, has at most two solutions\[6\].

(ii) Let $m > 1$ in (3). We start by writing

$$4q^m - 9 = 4q^n - x^2 = A a^2.$$ (4)

Where $A \geq 1$ is an odd square free. Suppose $p$ divides $m$, where $m$ is odd and put $q_1 = q^{m/p}$, we get:

$$4q_1^p - 9 = Aa^2.$$ (5)

If $A = 1$, then $a^2 \equiv -1 (\text{mod } 4)$, but $-1$ is not quadratic residue modulo 4, therefore $A \neq 1$.

Now if $A = 3$, then $q = 3$, and dividing equation (4) by 3, we get the equation

$$y^2 = 4q^{n-2} - 4q^{m-2} + 1,$$

which have been solved by Luca\[7\], and the only solution in our case ($q = 3$) is $n = 7$, $m = 3$ and $y = 31$, so $x = 93$. Also Luca refer to the case $n-2 = 2(m-2)$ as the trivial solution of this equation, and this will give us the solution $x = 2.3^{m-1} - 3$, as desired.

Hence we shall suppose that $q_1 \geq 5$, therefore $A \geq 5$ and $A \not \equiv 0 (\text{mod } 3)$. We write (5) as
Suppose $\langle q \rangle = \pi \bar{\pi}$, where $\pi$ is a prime ideal, therefore the two algebraic integers appearing in the right-hand side of (6) are coprime in the ring $Q(\sqrt{-A})$. Then

$$\pi^p \bar{\pi}^p = \left[ \frac{3 + \sqrt{-A}a}{2} \right] \left[ \frac{3 - \sqrt{-A}a}{2} \right].$$

This implies that

$$\pi^p = \left[ \frac{3 + \sqrt{-A}a}{2} \right] \text{ and } \bar{\pi}^p = \left[ \frac{3 - \sqrt{-A}a}{2} \right].$$

So $\pi^p$ is a principal ideal which implies $O(\pi) \mid p$, hence $\pi$ is a principal ideal.

Let $z = \frac{c + b\sqrt{-A}}{2}$, where $c \equiv b (\text{mod} \ 2)$, is a generator of $\pi$ then we get

$$\langle q \rangle = \langle z \rangle \cdot \langle \bar{z} \rangle$$

and

$$\langle z^p \rangle = \left[ \frac{3 + \sqrt{-A}a}{2} \right], \langle \bar{z}^p \rangle = \left[ \frac{3 - \sqrt{-A}a}{2} \right].$$

Since the units in the field $Q(\sqrt{-A})$ are $\pm 1$, therefore

$$\pm z^p = \frac{3 + \sqrt{-A}a}{2}, \quad \pm \bar{z}^p = \frac{3 - \sqrt{-A}a}{2}.$$

Hence

$$\frac{u_{2p}}{u_p} = z^p + \bar{z}^p = \pm 3.$$  \hspace{1cm} (7)

From (7) we get that $u_{2p} = \pm 3 u_p$ which implies that $u_{2p}$ has no primitive divisors.
Let \( p = 3 \) in equation (7), then we get

\[
\pm 3 = z^p + \overline{z}^p = \frac{c^3 - 3Ac^2}{4}.
\]

Or

\[
\pm 12 = c(c^2 - 3Ab^2)
\]  

(8)

If \( c \) is even, then so \( b \) is even, and in this case the right-hand side of (8) is a multiple of 8, which is impossible. Thus \( c \) is an odd divisor of 12, therefore \( c = \pm 3, \pm 1 \). From equation (8) we now conclude that \( 3Ab^2 = 5, -11, \pm 13 \), which is obviously impossible.

Assume now that \( p \geq 5 \), in this case, \( u_{2p} \) has no primitive divisors. From Table 2 and 3 in [5] and a few exceptional values of \( z \). None of the exceptional Lehmer terms from that Table leads to a value of \( z \in \mathbb{Q}(\sqrt{-A}) \). Thus equation (3) has no solutions when \( m \) is odd and \( q \geq 5 \).

Now let us suppose that \( m \) is even, say \( m = 2k \), and \( k \) is a positive integer. From equation (4), we get \( x^2 - 9 = 4q^n - 4q^{2k} \), which implies that

\[
\frac{x + 3}{2}, \frac{x - 3}{2} = q^{2k} (q^{n-2k} - 1).
\]

Since the two factors in the left hand side are coprime we get \( q = 2 \) and \( m = 2 \). Substituting in (4) we find the famous equation of Ramanujan \( x^2 = 2^{n+2} - 7 \), which has only the following solutions [2]

\[
(x, n) = (1, 1), (3, 2), (5, 3), (11, 5), (181, 13).
\]

This concludes the proof.

References


دراسة المعادلة الديوفنتية

فدوى أبو مريفة، و أمل الرادش

جامعة الرياض للبنات، الرياض، المملكة العربية السعودية

abumuriefah@yahoo.com

الخلاص. في هذا البحث درسنا المعادلة الديوفنتية:

\[ x^2 = 4q^n - 4q^m + 9 \]

حيث عدد أولي و 0 \( n > m \geq 0 \) أعداد صحيحة و قدمنا حلاً. \( m \geq 0 \) كاملاً عندما