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Product Formula for Imaginary Resolvents and its Application

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ABSTRACT. In this paper we study a product formula for imaginary resolvents of the Schrödinger operator $-\Delta + q$ as well as elliptic differential operators (generalized Schrödinger operators) of a complex Hilbert space H under the conditions [q], $[a_k]$ and $[b_k]$.

The resolvents and related functions are given by an integral kernel, which is bounded by a convolution with a radial decreasing L_1- function. We examine the weakest conditions under which the Schrödinger equation

$$i \frac{\partial u}{\partial t} = Ku \ , \text{ with } u(x,0) = v(x) \ , \ V \in D(K)$$

and K the realization in H of the Schrödinger operator (and the generalized Schrödinger operator), can be solved. The solution is represented as a sequential limit of finite dimensional integrals involving the kernel of the imaginary resolvent of $(-\Delta)$ and $(\sum_{k \leq t} a_k D^k)$.

Introduction

Trotter[1], Kato[2,3] and others have proved the existence of the limit

$$\lim_{n \to \infty} \left(e^{-i(t/n) T} e^{i(t/n) S} \right)^n \ , \text{ where}$$

T and S are self-adjoint operators. If T + S is essentially self-adjoint, then T + S has a unique self-adjoint extension K and by Trotter's theorem, we have:

$$\lim_{n \to \infty} \left(e^{-i(t/n) T} . e^{-i(t/n) S} \right)^n = e^{-itK} \tag{1}$$

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which is known as "Trotter's formula". Under certain conditions on S Kato[4], using Chernoff's lemma, has proved the existence of this formula in the special case of T = − Δ and S is a complex-valued measurable function. A generalization of Kato's theorem has been given by Barry Simon. These results can be applied to great variety of operators. Trotter's formula has also been verified by Kato[3] when T and S are nonnegative self-adjoint operators in a Hilbert space.

Lapidus[4] has shown that if the unitary groups generated by T and S are replaced by their associated imaginary resolvents, the corresponding product formula holds for a pair of nonnegative self-adjoint operators T and S, *i.e.:*

$$\lim_{n\to\infty} \left([I + i(t/n)T]^{-1} \, [I + i(t/n)S]^{-1} \right)^n = e^{-itk} \tag{2}$$

where K denotes the form sum of T and S.

By a somewhat different method Lapidus[4] obtained the product formula (2) for self-adjoint operators T and S, where T is nonnegative and it is essential that S has an unbounded negative part.

It is noteworthy that Kato[2,3] was interested in the behaviour of semigroups and not that of the resolvents. Furthermore Lapidus was interested in the behaviour of the resolvents of S, T and K = T + S. The functions $(z - A)^{-1}$ and exp (−tA) are related and many properties of one can be translated into appropriate properties for the other. However, for Schrödinger operators the study of semigroup precedes the study of the resolvent. Here we shall adopt and use the "Feynman integral" to search for the solution of the Schrödinger equation. As a matter of fact the resolvents kernels of (−Δ) and of elliptic differential operators and some other related functions have been studied in Gurarie and Gurarie and Kon[5,6].

In this paper we explore possible extensions of Lapidus's results to elliptic differential operators (generalized Schrödinger operators).

First we recall some notations and definitions as follows: The space $L_p(R^m)$ is defined to be the set of measurable functions f such that $\|f\|_p = (\int |f|^p \mu)^{1/p} < \infty$, $1 \leq p < \infty$ where μ will be the Lebesgue measure and R^m is m-dimensional real space. $L_{p,loc}(R^m)$ is the set of functions which lie in $L_p(W)$ for each compact $W \subset R^m$. $C_o^\infty(R^m)$ denotes, the space of infinitely many times differential functions with compact support. $W_{p,r}(R^m)$ denotes the set of functions f such that for $O \leq |k| \leq r$ all the weak derivatives $D^k f$ exist and are in $L_p(R^m)$, and equip $W_{p,r}(R^m)$ with a scalar product and norm as follows:

$$\langle f, g \rangle_r = \sum_{|k|\leq r} \int D^k f \, \overline{D^k g} \, dx, \quad \|f\|_r^2 = \sum_{|k|\leq r} \int |D^k f|^2 \, dx.$$

$W_{p,r}(R^m)$ is called a Sobolev space of order r. D(T) denotes the domain of the operator T. A multi-index k is an n-tuple $(k_1, k_2, \ldots, k_n)$ of non-negative integers. We write $K = k_1 + \ldots + k_n$. For $x \in R^m$,

$x^k = x_1^{k_1} \quad x_m^{k_m}$ and $D^k = D_1^{k_1} \quad D_m^{k_m}$, where $D_j = \frac{\partial}{\partial x_j}$

The operator $T_p = \sum_{|k|=r} a_k D^k$ is known as the principle part of $T = \sum_{|k|\leq r} a_k D^k$.

A real-valued measurable function q on R^m is to lie in Q_m if and only if

i) $\lim_{a\to o} (\sup_x \int_{|x-y|\leq a} |x-y|^{2-m} |q(y)| d^m y) = o$, if $m \geq 3$

ii) $\lim_{a\to o} (\sup_x \int_{|x-y|\leq a} \ln(|x-y|^{-1}) |q(y)| d^2 y) = o$, if $m = 2$

iii) $\sup_x \int_{|x-y|\leq 1} |q(y)|\, dy < \infty$, if $m = 1$.

The operator B is said to be A-bounded if $D(A) \subset D(B)$ and there exist $b \geq o$ and $a \geq o$ such that $\|Bf\| \geq a\|f\| + b\,\|Af\|$ (*), for all $f \in D(A)$. The infimum of all $b \geq o$ for which an $a \geq o$ exist such that (*) holds is called the A-bound of B.

I. Schrödinger Operator $(-\Delta + q)$

In this section we study the following example:

Let $T = -\Delta$ be the negative Laplacian operator in $L_2(R^m)$, $m \geq 3$ and let S be the multiplication operator on $L_2(R^m)$ with a real-valued function q with the following condition [q] :

$$[q] \qquad q_+ := \max(q,o) \quad , \quad q_+ \in L_{1,loc}(R^m)$$
$$q_- := \max(-q,o) \quad , \quad q_- \in Q_m.$$

To prove that (2) holds for this example we use theorem 1 in Lapidus[7]. This theorem is applicable if we show that the operator T + S is essentially self-adjoint on $D(T) \cap D(S)$ with $T = -\Delta$, S is a multiplication operator by q. In this case there exists a unique self-adjoint extension of T + S which coincides with the form sum $T \dotplus S$, *i.e.* the form sum $T \dotplus S$ is the realization of the Schrödinger operator T + S. To prove the existence of a unique self-adjoint extension $-\Delta + q$ we show that $-\Delta + q$ is essentially self-adjoint. For that we prove that q_- is form- $(-\Delta)$- bounded with $(-\Delta)$ form-bound < 1, $D(-\Delta) \subset D(q_-)$, (*see,* Lapidus[7]).

For that we have to show that:

$$\|\, |q_-|^{1/2} f\|^2 = |t(f,f)| \leq a\,((f,-\Delta) + z^2 \|f\|^2) = a\,(\|(-\Delta + z^2)^{1/2} f\|^2) ,$$

for arbitrary small $a > o$ if z^2 is chosen appropriately. Thus it suffices to show that $\|W_z\| \to o$ as $z \to \infty$, where

$$W_z = |q_-|^{1/2} (-\Delta + z^2)^{-1/2} \qquad \text{(see Hempel[8])}.$$

Since $\|W_z\| = \|W_z W_z^*\|^{1/2}$, it suffices to show that $\|W_z W_z^*\| \to o$, as $z \to \infty$.

It is clear that $W_z W_z^* \supset |q_-|^{1/2} (-\Delta + z^2)^{-1} |q_-|^{1/2}$, where $(-\Delta + z^2)^{-1}$ is an integral operator with the kernel G(r,z)

and

$$|G(r,z)| \leqslant \begin{cases} c(m)\, r^{2-m} \\ c(m)\, r^{-\frac{m-1}{2}} |z|^{\frac{m-3}{2}} \end{cases}$$

Now, for $u \in D(|q_-|^{1/2})$ let us define

$$g(y) = \int G(r,z)\, |q_-(x)|^{1/2} u(x)\, dx.$$

Then we have

$$\|W_z W_z^* u\| = \||q_-|^{1/2} g\|^2 = \int |q_-(y)| |g(y)|^2 dy = \iint |G(r,z)|\, |q_-(x)|\, |u(x)|^2 |q_-(y)|\, dx\, dy \leqslant$$

$$\sup_{y \in R^m} \int |G(r,z)|\, |q_-(x)|\, dx \iint |q_-(y)|\, |G(r,z)|\, |u(x)|^2\, dx\, dy$$

$$= c_z \iint |q_-(y)|\, |G(r,z)|\, |u(x)|^2\, dx\, dy, \text{ where}$$

$$c_z = \sup_{y \in R^m} \int |G(r,z)|\, |q_-(x)|\, dx.$$

Let $c_z = I_1 + I_2$, then

$$I_1 = \sup_{y \in R^m} \int_{|x-y|<|z|^{-1}} |G(r,z)|\, |q_-(x)|\, dx$$

and

$$I_2 = \sup_{y \in R^m} \int_{|x-y|>|z|^{-1}} |G(r,z)|\, |q_-(x)|\, dx \quad .$$

By using the estimation of the function G(r,z) we get

$$\int_{|x-y|<|z|^{-1}} |G(|x-y|, z)|\, |q_-(x)|\, dx \leqslant$$

$$c(m) \int_{|x-y|<|z|^{-1}} |x-y|^{2-m}\, |q_-(x)|\, dx$$

The condition [q] implies that $q_- \in Q_m$, $m \geqslant 3$, *i.e.*

$$\int_{|x-y|<|z|^{-1}} |x-y|^{2-m}\, |q_-(x)|\, dx \;\to o, \text{ as } z \to \infty$$

Hence $I_1 \to o$, as $z \to \infty$.

Also by using the estimation of the function G(r,z) and the condition [q] we can show that $I_2 \to o$, as $z \to \infty$. Therefore $c_z \to o$, as $z \to \infty$. This implies that:

$$\|W_z W_z^*\| \to o, \text{ as } z \to \infty, \textit{ i.e. } \|W_z\| \to o, \text{ as } z \to \infty.$$

Hence q_- is $-\Delta$ −form bounded with relative bound < 1. This proves that the operator $-\Delta + q$ is essentially self-adjoint. Thus there exists a self-adjoint extension of $-\Delta + q$ which we denote by K.

Hence by the theorem (1) mentioned by Lapidus[7],

$$\lim_{n\to\infty} ([I + i(t/n)(-\Delta)]^{-1} \; [I + i(t/n)q]^{-1})^n \; f = e^{-itK} f,$$

for all $f \in H$, uniformly in t on bounded subsets of R.

In order to find the unique solution of the Schrödinger equation

$$: \frac{\partial u}{\partial t} = Ku \;,\; \text{with } u(x,o) = v(x) \;,\; v \in D(K) \text{ and } K = -\Delta + q \tag{3}$$

we apply theorem (2) by Lapidus[7] and the estimations of the kernels of the resolvents of $-\Delta$.

The unique solution of (3) is given by:

$$u(x,t) = (e^{-itK}v)\;(x) \;,\; t \in R, x \in R^m.$$

Since $q_+ \in L_{1,loc}(R^m)$ and q_- is $-\Delta-$ form bounded with relative bound < 1, the modified Feynman integral converges and theorem (2) mentioned by Lapidus[7] holds. Therefore the solution of (3) is represented by a modified Feynman path integral as follows:

For all $v \in L_2(R^m)$ and almost every $x \in R^m$

$$(e^{-itK} v)\;(x) = \lim_{n\to\infty} \int_{R^m} \cdots \int_{R^m} F_n(x_o ,\ldots, x_n,t) \tag{4}$$

$$\times \prod_{j=1}^{n} (1 + i(t/n)\, q(x_j)\,)^{-1}\, v(x_n)\; dx_1 \quad dx_n \;,$$

where we have set $x_o = x$ and $F_n(x_o ,\ldots, x_n,t)$ is the nth iterated kernel of the convolution operator $[I - i(t/n)\,\Delta]^{-1}$;

$$F_n\,(x_o \quad , x_n,t) = \prod_{j=1}^{n} G(x_{j-1},\, x_y;\, t/n) \tag{5}$$

The convergence in (4) holds in $L_2(R^m)$ and is uniform in t on bounded subset of R. The function $G(x_{j-1} , x_j ; t/n)$, $m \neq 3$ turns out to be an expression in terms of the Bessel's functions of 3*rd* order and it is given by:

$$G(x_j \,,\, x_{j-1} \,,\, t/n) = (i/4 \, \frac{t/n}{2\,\pi |\, x_j - x_{j-1}|})^{m/2-1}\, H^{(1)}_{m/2-1}\,(x_j \,,\, x_{j-1} \,,\, t/n) \tag{6}$$

where $H^{(1)}_{m/2-1}\,(x_j \,,\, x_{j-} \quad , t/n)$ is the Bessel's function of 3*rd* order.

Therefore from (5) and (6),

$$F_n(x_O, \ldots, x_n, t) = \prod_{j=1}^{n} \left(i/4 \frac{t}{2\pi n |x_j - x_{j-1}|}\right)^{m/2} H^{(1)}_{\frac{m}{2}} (x_j, x_{j-1}, \frac{t}{n})$$

$$= \left(\frac{i t}{8 \pi n}\right)^{(m/2-1)n} \prod_{j=1}^{n} |x_j - x_{j-1}|^{1-\frac{m}{2}} H^{(1)}_{\frac{m}{2}-1} (x_j, x_{j-1}, \frac{t}{n}) \tag{7}$$

Substitute (7) in (4); we get

$$(e^{-itK} v)(x) = \lim_{n\to\infty} \int_{R^m} \cdots \int_{R^m} \left(\frac{i t}{8 \pi n}\right)^{(\frac{m}{2}-1)n} \prod_{j=1}^{n} |x_j - x_{j-1}|^{(1-m/2)}$$

$$\times H^{(1)}_{\frac{m}{2}-} (x_j, x_{j-1}, t/n)\ (1 + i(t/n)\, q(x_j))^{-1} v(x_n)\ dx_1 \cdots dx_n.$$

Let us consider a continuous path w connecting x_o to x_n in time t such that $w(o) = x_o$, $w(t) = x_n$ and the x_j lie in w. Hence $\prod_{i=1}^{n} (1 + i\,(t/n)\, q(x_j))^{-1}$ is an approximation for $\exp(-i \int_o^t q(\dot{w}(s))\, ds)$

This implies that

$$(e^{-itK} v)(\ ) = \int_{R^m} \cdots \int_{R^m} \left(\frac{i t}{8 \pi n}\right)^{(\frac{m}{2}-1)n} \prod_{j=1}^{n} |x_j - x_{j-1}|^{(1-m/2)}$$

$$\times H^{(1)}_{\frac{m}{2}-1} (x_j, x_{j-1}, t/n)\, v(x_n))\ e^{-i\int_o^t q(w(s))ds}\, dx \cdots dx_n \tag{8}$$

Remark: We note that the radial function $H^{(1)}_d(z)$ can be taken to be

$$H^{(1)}_d(z) = \begin{cases} |z|^{-d} & , \text{ if } o < |z| \leqslant 1,\ d > o \\ |z|^{-1/2} & , \text{ if } |z| > 1 \end{cases}$$

Hence the Schrödinger equation can be solved under the weakest condition [q] on q. The unique solution is given by (8). Also under the condition [q] the product formula (2) holds and hence the result.

II. Elliptic Differential Operator

In this section we consider elliptic operator $T = A + B$ on R^m under the following conditions $[a_k]$ and $[b_k]$:

$[a_k]$: T whose leading part A, where $A = \sum_{k \leq r} a_k(x)D^k$ is uniformly elliptic differential operator, *i.e.* the leading symbol $a(x,y) = \sum a_k(x)y^k$ satisfies the condition, $c_1 |y|^r \leq a(x,y) \leq c_2 |y|^r$ uniformly in $x \in R^m$. We assume that the coefficients $a_k(x)$ are sufficiently smooth and bounded with sufficient number of their derivatives.

Now the conditions on B are:

$[b_k]$: The operator $B = \sum b_k(x)D^k$ (order B $\leq$ order A), have coefficients $b_k \in L_{l_k, loc}(\mathbb{R}^m)$. We introduce for each term b_kD^k, its "fractional order", $s = \frac{n}{l_k} +$ $|k| \leq r$. (this condition is needed to have B bounded relative to A). For higher-order coefficients b_k we assume that $\sum_k \|b_k\| < 1$. Finally let $b_k \in L_{l_k} + L_\infty$

To prove the existence of the product formula for the operators A and B under the above conditions $[a_k]$ and $[b_k]$ we show that there exists a self-adjoint extension K of A + B as follows:

each term of the operator $B(z - A)^{-1} = \sum_k b_kD^k(z - A)^{-1}$ is composed of two operators T_1 and T_2, where T_1 is a multiplication operator with b_k and T_2 is a convulution with kernel $E_s(x) = F^{-1}(z^k(z - a(z))^{-1})$. Gurarie[5] and Gurarie and Kon[6] have proved the following estimation:

$$\|B(z - A)^{-1}\| \leq c(\theta)\, \rho^{d/(r-1)} \qquad (9),$$

$$c(\theta) = o|\theta|^{-\infty} (\infty > O) \ , \ z = \rho e^{i\theta},$$

taking the following considerations:

1) The radial function $H_{\mu,\beta}$ is given by:

$$H_{\mu,\beta}(z) = \begin{cases} |z|^{-\mu} & , \quad |z| \leq 1 \\ |z|^{-\beta} & \quad |z| > 1 \ , \end{cases}$$

where $-\infty < \beta < \infty$ and μ is the degree of smoothness of a(x,y) in y at o.

2) For each term b_kD^k the "fractional order" condition is imposed.

3) The Leibnitz's rule has been used in the sense that,

$$D^k(z - A)^{-1} f = \sum_{o \leq j \leq k} \binom{k}{j} ((z - a)^{-1})^{k-j} D^j f$$

4) The iterated chain rule for derivatives of $(z - a)^{-1}$ has been used,

$$\partial_x^j (z - a)^{-1} = \sum c_{j^1 \dots j^l} (y - a)^{-l-1} \prod_1^l \partial_x^{j^l} a \ ,$$

$1 \leq l \leq |j|$, $j^1 + \quad + j^l = j$, the summation being taken over all partitions of j into the sum of multiindeces $j^1 \dots j^l$ and $c_{j^1 \dots j^l}$ being certain universal combinatorial coefficients.

As a corollary of this estimation (9) we conclude the following:

1) a priori estimates for the operators A and B is

$$\|Bf\|_p \leq \epsilon \|Af\|_p + \lambda_\epsilon \|f\|_p \qquad , \quad 1 \leq p \leq \min l_k \tag{10}$$

for all f in D(A) in L_p, $o < \epsilon = c_p \Sigma \|b_k\|$, $\lambda_\epsilon > o$.

2) the fact that $D(B) \supset D(A)$ implies that $D(A + B) = D(A)$.

With a appropriate choice for c_p and $\Sigma\|b_k\| < 1$ (see the condition $[b_k]$), a priori estimates (10) is used along with Kato-Rellich theorem to prove the essential self-adjointness of the operator A + B on D(A), if A + B is formally symmetric.

Hence the operator A + B has a unique self-adjoint extension K which coincides with the realization of A + B.

Consequently and by use of theorem 1 due to Lapidus[7] we conclude that

$$\lim_{n \to \infty} ([I + i(t/n)A]^{-1} \; [I + i(t/n)B]^{-1})^n = e^{-itk} \; ,$$

uniformly in t on bounded subsets of R.

As for the solution of the Schrödinger equation (3) with K = A + B we use the theorem (2) in Lapidus[7] and the estimations of the radial bounds for the resolvents of the operator A (see[5,6]).

The unique solution of the Schröndinger equation is given by:

$$u(x,t) = (e^{-itK}v)(x), \qquad t \in R, x \in R^m.$$

By the theorem (2) in Lapidus[7] we can express this solution as follows:

$$(e^{-itk}v)\;(x) = \lim_{n\to\infty} \int_{R^m} \dots \int_{R^m} F_n(x_O, x_1, \dots, x_n, t)$$

$$\times \prod_{j=1}^{n} (1 + i(t/n)\, B(x_j))^{-1} \; v(x_n)\, dx_1 \quad dx_n \; ,$$

for all $v \in H$ and almost every $x \in R^m$, where $x = x_o$ and $F_n(x_o , \quad , x_n ; t)$ is the *nth* iterated kernel of the convolution operator $[I - i(t/n)A]^{-1}$,

$$F_n(x_o, \dots, x_n \; ; t) = \int_{j=1} G(x_{j-1} , x_j \; ; t/n) \; . \tag{12}$$

The convergence in (12) holds in $L_2(R^m) = H$ and is uniform in t on bounded subsets of R.

The estimations of the function $G(x_{j-1} , x_j ; (t/n))$ due to Gurarie[5] and Gurarie and Kon[6] are given by:

$$|G(x_{j-1}, x_j ; z)| \leq C\,|z|^{m/r} - \begin{cases} (|z|^{1/r}\,|x_j \quad - x_j|)^{-m+r} \; ; & |x_{j-1} - x_j| \leq 1 \\ e^{-r\,\mathrm{Im}^r\sqrt{z}}\;|x_{j-1} - x_j| \; ; & |x_{j-1} - x_j| < 1 \end{cases}$$

where we set $z^2 = n/it$, $\mathrm{Im}\, z > o$.

Given a continuous path w connecting x_o to x_n in time t such that $w(o) = x_o$, $w(t) = x_n$ and the x_j lie in w. Hence

$\prod_{j=1}^{n} (1 + i(t/n)\,B(x_j))^{-1}$ is an approximation for $e^{-i\int_o^t B(w(s))ds}$

Since all paths are continuous for each x,

$$v(x(t)) \prod_{j=1}^{n} (1 + i(t/n)\,B(x_j))^{-1} \rightarrow v(w(t))\, e^{-i\int_o^t B(w(s))\,ds}$$

Substitute (13) in (12) we get

$$F_n(x_O , \quad x_n ; t) \leq \prod_{j=1}^{n} C\,|z|^{m/r} - \begin{cases} (|z|^{1/r}\,|x_{j-1} - x_j|)^{-m+r} \; ; & |x_{j-1} - x_j| \leq \\ e^{-r\,\mathrm{Imr}\sqrt{z}}\;|x_{j-1} - x_j| \; ; & |x_{j-1} - x_j| > 1 , \end{cases} \qquad (15)$$

where $z^2 = n/it$, $\mathrm{I\,m}\, z > o$.

From (14) and (15) in (11) we obtain the solution of the Schrödinger equation (3) under the conditions $[a_k]$ and $[b_k]$ on the coefficients of the operators A and B and hence the result.

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قاعـــدة حاصل الضرب للمؤثرات الحَـــلِّيـــة التخيليـــة وتطبـيـقها

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نبحث تحقيق قاعدة حاصل الضرب للمؤثرات الحلية التخيلية

$$\lim_{n\to\infty} ([I + i(t/n)T]^{-1} \ [I + i(t/n)S]^{-1})^n = e^{-itK}$$

على فراغ هلبرت H في الحالات التالية :

١ - مؤثر شرودنجر $q + \Delta -$ تحت الشرط [q] المذكور في الفصل الأول

٢ - المؤثر التفاضلي التناقصي (مؤثرات شرودنجر العامة)

$$A = \sum_k a_k D^k \qquad B = \sum_j b_j D^j \qquad (\text{order } A \geqslant \text{order } B).$$

تحت الشروط $[a_k]$, $[b_j]$ المذكورة بالفصل الثاني

لبرهنة صحة قاعدة حاصل الضرب للمؤثرات الحلية التخيلية سوف نثبت أن المؤثر $q + \Delta -$ مترافق ذاتيا أسـاسيا على تقاطع نطاق المؤثر $\Delta -$ ونطاق المؤثر الضربي بوساطة الدالة $q(x)$ وفي الحالة الثانية يبرهن وجود مؤثر K بحيث يكون امتداد مترافق ذاتي $A + B$ بعد ذلك نستخدم نظرية (1) في [6] ثم ندرس في كلتا الحالتين حل معادلة شرودنجر

$$i \frac{\partial u}{\partial t} = Ku \qquad u(x,o) = v(x) \qquad v \in (D(K)$$

باستخدام نظرية (2) في [6] نجد أن الحل لمعادلة شرودنجر هو نهاية متعاقبة لتكاملات (ذات أبعاد محدودة) تحتوى على نواة المؤثرات الحلية التخيلية لكل من $\Delta -$,A .

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