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A New Scattering Theory Related Bessel Function Result

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ABSTRACT. Some scattering-theory related Bessel function results are reviewed, and a new expression involving the squares of Bessel fnnctions is then obtained. The new expression is

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l\left(\rho^{l+1}j_l(\rho)\right)^2 = 2\rho^{l+1}j_l(2\rho), \text{ where } \frac{d}{d\rho^2}=\frac{d}{2\rho d\rho}.$$

1. Review

Consider a potential $V_l(r)$ where the potential may depend on the angular momentum l, but not the momentum, and where the potential is sufficiently weak. In this case one can use the partial-wave Born approximation namely:

$$\tan\delta_l(k) = \frac{2Mk}{\hbar^2}\int_0^\infty V_l(r)\, j_l^2(kr) r^2 dr, \tag{1}$$

where $E = k^2\hbar^2/(2M)$, and M is the mass of the scattered particle, and $\delta_l(k)$ the resulting phase shift.

If one knows $V_l(r)$, one can, with the help of eq. (1), find the corresponding phase shifts $\delta_l(k)$, which are k, or equivalently E dependent, and hence the differential and total scross sections. For example, one has the standard formula:

$$\sigma(k) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k). \tag{2}$$

Let us now turn our attention to the inverse problem, in this approximation. In other words, suppose we know $\delta_l(k)$, can we uniquely determine the potential $V_l(r)$? For an exact solution of this inverse prohlem one must, as is known, solve the Gel'fand-Levitan Marchenko equation[1,2].

Looking carefully at eq. (1) one realizes that if one finds the inverse function $g_{ll}(kr')$, of $j_l(kr)j_l(kr) = j_l^2(kr)$ where

$$\int_0^\infty j_l^2(kr)\, g_{ll}(kr')dk = \delta(r-r'). \tag{3}$$

one can formally obtain $V_l(r)$ and hence any integral of $V_l(r)$ in the partial-wave Born approximation.
This is accomplished by pre-multiplying eq. (1) by $\{g_{ll}(kr')\}/k$, integrating over dk and interchanging the k and r integrations. Thus

$$\int_0^\infty \frac{g_{ll}(kr')\tan\delta_l(k)}{k}dk = -\frac{2M}{\hbar^2}\int_0^\infty j_l^2(kr)g_{ll}(kr')dkV_l(r)r^2dr$$
$$= -\frac{2M}{\hbar^2}\int_0^\infty \delta(r-r')V_l(r)^2dr = -\frac{2M}{\hbar^2}V_l(r')r'^2.$$

Hence:

$$V_l(r) = -\frac{\hbar^2}{2Mr^2}\int_0^\infty \frac{\tan\delta_l(k)g_{ll}(kr)}{k}dk \tag{4}$$

$$\int_0^\infty V_l(r)f(r)dr = -\frac{\hbar^2}{2M}\int_0^\infty\left\{\int_0^\infty \frac{\tan\delta_l(k)g_{ll}(kr)}{k}dk\right\}\frac{f(r)}{r^2}dr \tag{5}$$

where $f(r)$ is a well-behaved funetion of r.

One has thus obtained the potential $V_l(r)$ and arbitrary matrix elements of this potential in the partial-wave Born approximation, assuming the corresponding phase shifts are known. The crucial eqnations in this context being eqs. (3) and (4).

I investigated this possibility3, and concluded that the required inverse function of $j_l^2(kr)$ was

$$g_{ll}(\rho) = -\frac{4\rho^2}{\pi\left(l+\frac{1}{2}\right)}\,{}_1F_2\left(\frac{3}{2};\frac{1}{2}-l,l+\frac{3}{2};-\rho^2\right). \tag{6}$$

By similar techniques I also obtained the "inverse" functions $g_{l,l+1}(kr')$ for $j_l(kr)j_{l+1}(kr)$, and $g_{l,l+2}(kr')$ for $j_l(kr)j_{l+2}(kr)$, the latter being useful if one has tensor forces:

$$g_{l,l+1}(\rho) = g_{l+1,l}(\rho) = -\frac{4\rho^3}{\pi\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)}\,{}_1F_2\left(\frac{3}{2};\frac{1}{2}-l,l+\frac{5}{2};-\rho^2\right),$$

$$g_{l+2,l}(\rho) = g_{l,l+2}(\rho) = -\frac{8\left(l+\frac{3}{2}\right)}{\pi}\rho^4\frac{d}{d\rho^2}\left\{\frac{1}{\rho^2}\,{}_1F_2\left(\frac{1}{2};-l-\frac{3}{2},l+\frac{3}{2};-\rho^2\right)\right\}. \tag{7}$$

A year later Sollfrey[4] obtained the following equivalent formula for $g_{ll'}(\rho)$ by using Mellin transforms:

$$g_{ll'}(\rho) = \frac{8\rho^2}{\pi}\frac{d}{d\rho}\{\rho^2 n_l(\rho)j_{l'}(\rho)\}. \tag{8}$$

More recently two new expressions[5,6] were obtained for $g_{ll}(\rho)$, namely:

$$g_{ll}(\rho) = -\frac{4}{\pi}\rho^{2l+2}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^l\frac{d}{d\rho}\left(\frac{d}{d\rho}\frac{1}{2\rho}\right)^l\sin(2\rho) \tag{9}$$

$$g_{ll}(\rho) = (-1)^{l+1}\frac{8\rho^2}{\pi}\left[\cos(2\rho) - 2l(l+1)\sum_{k=0}^{l}\frac{(-1)^k 2^k(l+k)!}{(k+1)!(l-k)!}\left\{\frac{j_k(2\rho)}{(2\rho)^k}\right\}\right]$$

$$= (-1)^{l+1}\left[g_{00}(\rho) - \frac{16l(l+1)\rho^2}{\pi}\sum_{k=0}^{l}\frac{(-1)^k 2^k (l+k)!}{(k+1)!(l-k)!}\left\{\frac{j_k(2\rho)}{(2\rho)^k}\right\}\right]. \qquad (10)$$

In Table 1 a few typical $g_{ll'}$'s and j_l's are listed. The Appendix lists other useful mathematical properties of the $g_{ll'}$'s.

Table (1). A few typical $g_{ll'}$'s and $j_l(\rho)$'s

l	l'	$g_{ll'}(\rho)$	$J_l(\rho)$
0	0	$-\frac{8\rho^2}{\pi}\cos(2\rho)$	$\frac{1}{\rho}\sin(\rho)$
1	0	$-\frac{8\rho^2}{\pi}\left\{\left(1-\frac{1}{2\rho^2}\right)\sin(2\rho)+\frac{1}{\rho}\cos(2\rho)\right\}$	$\frac{1}{\rho^2}\sin(\rho)$
1	1	$\frac{8\rho^2}{\pi}\left\{\left(1-\frac{2}{\rho^2}\right)\cos(2\rho)+\left(\frac{1}{\rho^3}-\frac{2}{\rho}\right)\sin(2\rho)\right\}$	$-\frac{1}{\rho}\cos(\rho)$
2	0	$\frac{8\rho^2}{\pi}\left\{\left(1-\frac{9}{2\rho^2}\right)\cos(2\rho)+3\left(\frac{1}{\rho^4}-\frac{1}{\rho}\right)\sin(2\rho)\right\}$	$\left(\frac{3}{\rho^3}-\frac{1}{\rho}\right)\sin(\rho)$
2	2	$-\frac{8\rho^2}{\pi}\left\{\left(1-\frac{18}{\rho^2}+\frac{36}{\rho^4}\right)\cos(2\rho)+\left(-\frac{6}{\rho}+\frac{33}{\rho^3}-\frac{18}{\rho^5}\right)\sin(2\rho)\right\}$	$-\frac{3}{\rho^2}\cos(\rho)$

Examining this formalism more carefully one realizes that eq. (3) is almost, though not quite, correct. The exact result is:

$$\int_0^\infty j_l^2(kr)g_{ll}(kr')dk = \frac{r'^2}{r^2}\left\{\delta(r-r') + \delta(r+r') - (-1)^l 2\delta(r')\right\} \qquad (11)$$

where the $g_{ll'}(\rho)$ are defined in eqs (6) - (10). This means that in the Born approximation

$$\int_0^\infty \frac{g_{ll}(kr')\tan\delta_l(k)}{k}dk = -\frac{2M}{\hbar^2}\int_0^\infty\int_0^\infty j_l^2(kr)g_{ll}(kr')\,dk\,V_l(r)r^2dr$$

$$= -\frac{2M}{\hbar^2}\int_0^\infty \delta(r-r')V_l(r)r^2dr - \frac{2M}{\hbar^2}\int_0^\infty \delta(r+r')V_l(r)r^2dr + (-1)^l\frac{4M}{\hbar^2}\int_0^\infty V_l(r)dr\, r'^2\,\delta(r')$$

$$= -\frac{2M}{\hbar^2}V_l(r')r'^2 + (-1)^l\frac{4M}{\hbar^2}\int_0^\infty V_l(r)dr\, r'^2\,\delta(r'),$$

since $r, r' \geq 0$, i.e. rather than eq. (4), one has:

$$V_l(r) = -\frac{\hbar^2}{2Mr^2}\int_0^\infty \frac{g_{ll}(kr)\tan\delta_l(k)}{k}dk + (-1)^l 2\left\{\int_0^\infty V_l(r)dr\right\}\delta(r). \qquad (12)$$

The additional term $(-1)^l 2\left\{\int_0^\infty V_l(r)dr\right\}\delta(r)$ does not affect eq. (5), provided $f(r)$ vanishes at $r = 0$. This is the case for instance if $f(r) = r^4\exp(-r^2/2b^2)$.

An alternative approach to this problem that avoids the above complications at $r = 0$, involves using the subsequently derived expression[5]:

$$O(\rho)\left(\rho^{l+1}j_l(\rho)\right)^2 = \sin(2\rho), \qquad (13)$$

where

$$O(\rho)\equiv\left(\frac{d}{d\rho^2}\right)^l\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l,\quad \left(\frac{d}{d\rho^2}\right)^l\equiv\left(\frac{1}{2\rho}\frac{d}{d\rho}\right)^l.$$

Pre-multiplying both sides of eq. (1) by k^{2l+2} and then operating on the resulting expression with the operator $O(k)$ of eq. (13), (where if one writes $\rho = kr$, $O(k) = r^{4l+1}O(\rho)$), one obtains:

$$\left(\frac{d}{dk^2}\right)^l\frac{d}{dk}\left(\frac{d}{dk^2}\right)^l\left(k^{2l+2}\left\{-\frac{\hbar^2}{2M}\frac{\tan\delta_l(k)}{k}\right\}\right)=\int_0^\infty \sin(2kr)V_l(r)r^{2l+1}dr. \qquad (14)$$

which is just the Fourier-sine transform[7] of $(1/2)V_a(r)\, r^{2l+1}$.

Multiplying both sides of eq. (14) by $(4/\pi)\sin(2kr')$ and integrating over k one then obtains

$$V_l(r')=\frac{4}{\pi\, r'^{2l+1}}\int_0^\infty\left[\left(\frac{d}{dk^2}\right)^l\frac{d}{dk}\left(\frac{d}{dk^2}\right)^l\left(k^{2l+2}\left\{-\frac{\hbar^2}{M}\frac{\tan\delta_l(k)}{k}\right\}\right)\right]\sin(2kr')dk, \qquad (15)$$

since

$$\frac{4}{\pi}\int_0^\infty \sin(2kr')dk=\frac{1}{\pi}\int_0^\infty\{\cos 2k(r-r')-\cos 2k(r+r')\}d(2k)$$
$$=\delta(r-r')-\delta(r+r').$$

For the cases $l = 0, 1$, one thus has:

$$V_{l=0}(r')=\frac{4}{\pi\, r'}\int_0^\infty\left[\frac{d}{dk}\left(k^2\left\{-\frac{\hbar^2}{2M}\frac{\tan\delta_{l=0}(k)}{k}\right\}\right)\right]\sin(2kr')dk, \qquad (16)$$

$$V_{l=1}(r')=\frac{4}{\pi\, r'^3}\int_0^\infty\left[\left(\frac{d}{dk^2}\right)\frac{d}{dk}\left(\frac{d}{dk^2}\right)\left(k^4\left\{-\frac{\hbar^2}{2M}\frac{\tan\delta_{l=1}(k)}{k}\right\}\right)\right]\sin(2kr')dk, \qquad (17)$$

and so on for larger l.

Starting from eq. (15) one also obtains, for arbitrary $f(r)$:

$$\int_0^\infty V_l(r)f(r)dr=$$

$$\frac{4}{\pi}\int_0^\infty\int_0^\infty\frac{f(r)\sin(2kr)}{r^{2l+1}}dr\left[\left(\frac{d}{dk^2}\right)^l\frac{d}{dk}\left(\frac{d}{dk^2}\right)^l\left(k^{2l+2}\left\{-\frac{\hbar^2}{M}\frac{\tan\delta_l(k)}{k}\right\}\right)\right]dk. \qquad (18)$$

II. New Results

The spherical Bessel function $j_l(x)$ can be expressed as an infinite sum as follows:

$$j_l(x)=\sum_k^\infty\frac{(-1)^k 2^{2l+2k}(l+k)!}{k!(2l+2k+1)!}\left(\frac{x}{2}\right)^{l+2k}. \qquad (19)$$

From Appendix B of reference 5, we observe

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2 = 2^{2l+1}\sum_k^\infty \frac{(-1)^k 2^{2k}(l+k)!}{(2l+2k+1)!k!}\rho^{2l+2k+1}. \tag{20}$$

However, the rhs of this expression may be put in a more suggestive form:

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2 = 2\rho^{l+1}\sum_k^\infty \frac{(-1)^k 2^{2(k+1)}(l+k)!}{k!(2l+2k+1)!}\left(\frac{2\rho}{2}\right)^{l+2k}. \tag{21}$$

Comparing eqs (19) and (21) we thus obtain the simple new result:

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2 = 2\rho^{l+1} j_l(2\rho). \tag{22}$$

If one applies the standard results

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^m \left(\rho^{l+1} j_l(\rho)\right) = \rho^{l+1-m} j_{l-m}(\rho) \tag{23}$$

and

$$(-1)^m\left(\frac{1}{\rho}\frac{d}{d\rho}\right)^m \left(\frac{j_l(\rho)}{\rho^l}\right) = \frac{j_{l+m}(\rho)}{\rho^{l+m}}. \tag{24}$$

on eq. (21), one obtains additionally:

$$j_{l-m}(2\rho) = \frac{1}{2\rho^{l+1-m}}\left(\frac{d}{d\rho^2}\right)^m \frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2. \tag{25}$$

and

$$j_{l+m}(2\rho) = (-1)^m \frac{\rho^{l+m}}{2}\left(\frac{d}{d\rho^2}\right)^m \frac{1}{\rho^{2l+1}}\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2. \tag{26}$$

For the special case $m = l$, these reduce to:

$$j_0(2\rho) = \frac{1}{2\rho}\left(\frac{d}{d\rho^2}\right)^l \frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2. \tag{27}$$

and

$$j_{2l}(2\rho) = (-1)^l \frac{\rho^{2l}}{2}\left(\frac{d}{d\rho^2}\right)^l \frac{1}{\rho^{2l+1}}\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} j_l(\rho)\right)^2. \tag{28}$$

One notes that eq. (27) is equivalent to eq. (13). Thus, eq. (13) is merely a special case of the more general eq. (25), when $m = l$, which in turn stems from the new result of this paper, namely, eq. (22).

Analogously with eq. (22), for the spherical Neumann functions one has:

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left(\rho^{l+1} n_l(\rho)\right)^2 = -2\rho^{l+1} j_l(2\rho). \tag{29}$$

Combining eqs (22) and (29) one has the interesting differential equation:

$$\frac{d}{d\rho}\left(\frac{d}{d\rho^2}\right)^l \left[\left(\rho^{l+1} j_l(\rho)\right)^2 + \left(\rho^{l+1} n_l(\rho)\right)^2\right] = 0. \tag{30}$$

With the help of eq. (22) one may readily invert expression (1). Thus:

$$-\frac{\hbar^2}{4Mk^{l+1}}\frac{d}{dk}\left(\frac{d}{dk^2}\right)^l \left[k^{2l+1}\tan\delta_l(k)\right] = \int_0^\infty r^{l+2} V_l(r) j_l)2kr)dr, \tag{31}$$

and using the Hankel Transform[6] result,

$$\frac{2}{\pi}\int_0^\infty (xy)^2 j_l(xy) j(xy')dx = \delta(y-y'), \tag{32}$$

one has

$$r^l V_l(r) = \frac{-4\hbar^2}{M\pi}\int_0 \frac{1}{k^{l-1}}\left\{\frac{d}{dk}\left(\frac{d}{dk^2}\right)^l \left[k^{2l+1}\tan\delta_l(k)\right]\right\} j_l(2kr)dk. \tag{33}$$

Thus, for $l = 1$,

$$rV_1(r) = \frac{-4\hbar^2}{M\pi}\int_0^\infty \left\{\frac{d}{dk}\left(\frac{d}{dk^2}\right)\left[k^3\tan\delta_1(k)\right]\right\} j_1(2kr)dk. \tag{34}$$

One notes that only two differentiations of the phase shift are required here, as opposed to three in eq. (17) which is an improvement in the case the phase shifts are determined experimentally and every differentiation leads to more errors.

III. Conclusion

A new result, namely eq. (22) that relates spherical Bessel functions squared $j_l(\rho)^2$ to spherical Bessel functions of twice the argument $j_l(2\rho)$, is obtained, with an analogous expression for the spherical Neumann functions, namely eq. (29).

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Appendix

Starting from eq. (6) one eau obtain an integral expression for $g_{ll}(\rho)$, expressing it as a particular weighted integral over the Neumann function:

$$g_{ll}(\rho) = \frac{(-1)^l 16\rho^{l+3}\Gamma\left(\frac{1}{2}-l\right)}{\pi^{3/2}\left(l+\frac{1}{2}\right)B\left(\frac{3}{2},l\right)} \int_0^1 y^{l+3}\left(1-y^2\right)^{l-1} n_l(2\rho y)dy \qquad (l>0). \qquad (\text{A.1})$$

One can easily see from eqs (8), (9), or (10) that for large ρ

$$g_{ll}(\rho)_{\rho\to\infty} \frac{8(-1)^{l+1}}{\pi}\rho^2 \cos(2\rho), \qquad (\text{A.2})$$

while for small ρ one sees from Eq. (8) or (A.1) that

$$g_{ll}(\rho)_{\rho=0} - \frac{8\rho^2}{(2l+1)\pi}. \qquad (\text{A.3})$$

From eq. (8) one has

$$\begin{aligned}\int_0^\infty \frac{g_{ll}(xy)}{x^2y^2}dx &= \frac{1}{y}\frac{8}{\pi}x^2y^2 n_l(xy)\, j_l(xy)\Big|_0^\infty \\ &= -\lim_{x\to\infty}\frac{1}{y}\left[\frac{8}{\pi}\sin\left(xy-\frac{l\pi}{2}\right)\cos\left(xy-\frac{l\pi}{2}\right)\right] \\ &= -\frac{4}{\pi}\lim_{x\to\infty}\left[\frac{\sin(2xy-l\pi)}{y}\right] = (-1)^{l+1}\frac{4}{\pi}\lim_{2x\to\infty}\frac{\sin(2xy)}{y} \\ &= 4(-1)^{l+1}\delta(y). \end{aligned} \qquad (\text{A.4})$$

For the case[5] $f(r) = r^{2l}$ eq. (18) reduces to

$$\int_0^\infty V_l(r)r^{2l}dr = 2\int_0^\infty \left(\frac{d}{dk^2}\right)^l \frac{d}{dk}\left(\frac{d}{dk^2}\right)^l\left(k^{2l+2}\left\{-\frac{\hbar^2}{M}\frac{\tan\delta_l(k)}{k}\right\}\right)dk, \qquad (\text{A.5})$$

while[6] for $f(r) = r^{2l+1}$

$$\int_0^\infty V_l(r)r^{2l+1}dr = \frac{4}{\pi}\int_0^\infty \frac{1}{2k}\left(\frac{d}{dk^2}\right)^l \frac{d}{dk}\left(\frac{d}{dk^2}\right)^l\left(k^{2l+2}\left\{-\frac{\hbar^2}{M}\frac{\tan\delta_l(k)}{k}\right\}\right)dk. \qquad (\text{A.6})$$

نتيجة جديدة لنظرية التناثر باستخدام دالة بسل

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المستخلص. في هذا البحث نعرض بعض النتائج لنظرية التناثر باستخدام دالة بسل، حيث حصلنا على تمثيل جديد يحتوي على مربعات دالة بسل.