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COMMUTATIVITY OF RINGS WITH POLYNOMIAL CONSTRAINTS

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Abstract: The objective of this work is to establish the commutativity of rings with unity 1 and one-sided s-unital rings under each of the following properties: $y^r[x, y^m]x^s = x^p[x^n, y]x^q$ and $x^s[x, y^m]y^r = x^p[x^n, y]x^q$, where $r \geq 0, s \geq 0$ and $m > 1$ are fixed integers and for each x in R there exist integers $n = n(x) \geq 0, p = p(x) \geq 0, s = s(x) \geq 0$ and $q = q(x) \geq 0$ for every $y \in R$ such that, R has the property $Q(m)$, that is, $m[x, y] = 0$ implies that $[x, y] = 0$, for all $x, y \in R$. Further, we provide some counterexamples which show that the hypotheses of our theorems are not altogether superfluous. Finally, many well-known commutativity theorems become corollaries of our results.

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1. Introduction

Throughout, R will be an associative ring, $Z(R)$ the center of $R, C(R)$ the commutator ideal of $R, N(R)$ the set of all nilpotent elements of $R, N'(R)$ the set of all zero-divisors in $R, [x, y]$ the commutator $xy - yx$ of two elements x and y in $R, \mathbf{Z}[X, Y]$ the ring of polynomials in two commuting indeterminates and $\mathbf{Z} < X, Y >$ the ring of polynomials in two non-commuting indeterminates over the ring $\mathbf{Z}$ of integers. Following [2], a ring R is said to be left (resp. right) s-unital, if $x \in Rx$ (resp., $x \in xR$) for each element

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Theorem 2.2. *Let R be a ring with unity 1 satisfying the property (P_1). If R has the property $Q(m)$, then R is commutative.*

We begin with the following known results.

Lemma 2.1. (see [3, p. 221]) *If $[x, y]$ commutes with x, then for any positive integer k, $[x^k, y] = kx^{k-1}[x, y]$.*

Lemma 2.2. (see [4, Theorem]) *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with relatively prime integral coefficients. Then the following are equivalent:*

(i) For any ring satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(ii) For every prime p, $(GF(p))_2$ the ring of all 2×2 matrices over $GF(p)$, fails to satisfy $f = 0$.

Lemma 2.3. (see [11, Hauptsatz]) *Let R satisfy a polynomial identity of the form $[x, y] = p(x, y)$, where $p(X, Y)$ in $\mathbf{Z} < X, Y >$ has the following properties:*

(a) $p(X, Y)$ is in the kernel of the natural homomorphism from $\mathbf{Z} < X, Y >$ to $\mathbf{Z}[X, Y]$;

(b) each monomial of $p(X, Y)$ has total degree at least 3;

(c) each monomial of $p(X, Y)$ has X-degree at least 2, or each monomial of $p(X, Y)$ has Y-degree at least 2.

Then R is commutative.

Lemma 2.4. (see [10, Lemma 4]) *Let R be a ring with unity 1 and let $f : R \to R$ be any polynomial function of two variables with the property $f(x + 1, y) = f(x, y)$, for all x, y in R. If for all x, y in R there exists a positive integer $n = n(x, y)$ such that $x^n f(x, y) = 0$ (or $f(x, y)x^n = 0$), then necessarily $f(x, y) = 0$.*

We shall prove the following results.

Result 2.1. *Let R be a ring with 1 satisfying (P). Then $C(R) \subseteq N(R)$.*

Proof. Let R satisfy the property (P). We have

$$y^r[x, y^m]x^s = x^p[x^n, y]x^q. \tag{2.1}$$

Replacing y by $y + 1$ in (2.1), we get

$$(y + 1)^r[x, (y + 1)^m]x^s = x^p[x^n, y]x^q. \tag{2.2}$$

Combining (2.1) and (2.2),we get

$$((y + 1)^r[x, (y + 1)^m] - y^r[x, y^m])x^s = 0.$$

An application of the property $Q(m)$, we get

$$[x, a^{t-1}] = 0$$

that is, $a^{t-1} \in Z(R)$, which contradicts the minimality of t in (2.4). Hence $t = 1$ and $a \in Z(R)$, give the required result. □

Result 2.4. *Let R be a ring with unity 1 satisfying the property (P_1). If R has the property $Q(m)$, then $N(R) \subseteq Z(R)$.*

Proof. As in Result 2.1, similar arguments maybe used if R satisfies (P_1). □

Proof of Theorem 2.1. In view of Results 2.1 and 2.3, we get

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{2.7}$$

In view of (2.7), property (P) and by Lemma 2.1, we get

$$mx^s[x, y]y^{m+r-1} = n[x, y]x^{p+q+m-1}. \tag{2.8}$$

Replacing $1 + y$ for y in (2.8), we get

$$mx^s[x, y](1 + y)^{m+r-1} = n[x, y]x^{p+q+m-1}. \tag{2.9}$$

From (2.8) and (2.9), we get

$$mx^s[x, y]\{(1 + y)^{m+r-1} - y^{m+r-1}\} = 0, \text{ for all } x, y \in R.$$

Replacing $1 + x$ for x, by using Lemma 2.4 and the property $Q(m)$ in the last equation, we get

$$[x, y]\{(1 + y)^{m+r-1} - y^{m+r-1}\} = 0. \tag{2.10}$$

For $m + r = 2$ in (2.10), we get the commutativity of R.

For $m + r > 2$, (2.10) implies that $[x, y] = [x, y]f(y)$, for all x, y in R, and for some polynomial $f(Y)$ in $\mathbf{Z}[Y]$ all monomials of f have degree at least one. Hence R is commutative by Lemma 2.3. □

Proof of Theorem 2.2. Using Results 2.2, 2.4 and similar arguments with necessary variations in the proof of Theorem 2.1, we can get the required result. □

Remark 2.3. The following example strengthens the existence of the property $Q(m)$ in Theorems 2.1 and 2.2.

Example 2.1. Let $R = \begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$, where $\alpha, \beta, \gamma \in GF(4)$, the finite Galois field, be the set of all matrices. It is readily verified that R (with the usual matrix addition and multiplication) is a non-commutative local ring such that a ring R has the property P if and only if all its finitely generated subrings have P is called an F-property.

Lemma 3.1. (see [2, Proposition 1]) *Suppose that P is an H-property, and P' is an F-property. If every ring R with unity 1 having the property P has the property P', then every s-unital ring having P has P'.*

Remark 3.1. The results proved in the preceeding section can be automatically extended from a unital ring to s-unital rings by Lemma 3.1.

Indeed, we have

Theorem 3.1. *Let R be a left (resp. right) s-unital ring satisfying the property (P). Then R is commutative.*

Theorem 3.2. *Let R be a left (resp. right) s-unital ring satisfying the property (P_1). Then R is commutative.*

Remark 3.2. As a consequence of Theorems 3.1 and 3.2, we get the following corollary which includes [1, Theorems 1-4] and [7, Theorems 2 and 3] and [9, Theorem], [10, Theorem].

Corollary 3.1. *Let $m > 1, p, q, n, s$ and r be fixed non-negative integers and R a left (resp. right) s-unital ring satisfying (P) or (P_1). Then R is commutative in each of the following cases:*

(I) R has the property $Q(m)$;

(II) $n > 1$ and $m > 1$ are relatively prime integers at $s = 0$.

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with unity I, the identity matrix. Further, R satisfies

$$x^{48} \in Z(R) \quad \text{for all} \quad x \in R. \tag{2.11}$$

Now $N'(R)$ consists of all matrices x in R with zero diagonal elements, and thus, contains exactly 16 elements. For any $x \in N'(R), x^2 = 0$ and hence $x^{48} = 0 \in Z(R)$. The set $R|N'(R)$ is a multiplicative group of order 48 and hence $x^{48} = I \in Z(R)$ for all $x \in R|N'(R)$. In view of (2.11), it follows that R satisfies the properties (P) and (P_1) for the same m and n and for arbitrary non-negative integers p, q and r. This shows that the assumption R has the property $Q(m)$ in Theorems 2.1 and 2.2 cannot be eliminated.

Remark 2.4. The following result demonstrates that the conclusion of Theorems 2.1 and 2.2 are still valid, at $s = 0$, if the property "$Q(m)$" is replaced by the condition that "m and n are relatively prime positive integers."

Corollary 2.1. (see [7, Theorem 2]) *Let $m > 1$ and $r \geq 0$ be fixed integers and let R be a ring with unity 1 in which for every $x \in R$ there exist integers $n = n(x) > 1, p = p(x) \geq 0$ and $q = q(x) \geq 0$ such that m and n are relatively prime and R satisfies: $y^r[x, y^m] = \pm x^p[x^n, y]x^q$ or $[x, y^m]y^r = \pm x^p[x^n, y]x^q$. Then R is commutative.*

Remark 2.5. The following example shows that R is not commutative if "m and n are not relatively prime" in the hypothesis of above Corollary 2.1.

Example 2.2. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in GF(2) \right\}$. Then R is a non-commutative ring with unity 1 satisfying $y^r[x, y^4]x^s = x^p[x^4, y]x^q$ (or $x^s[x, y^4]y^r = x^p[x^4, y]x^q$), for any non-negative integers p, q, r and $s = 0$.

3. Extension to One-Sided s-Unital Rings

Since there are non-commutative rings with R^2 being central, neither (P) nor (P_1) guarantees the commutativity of arbitrary rings. Before we go ahead with our task, we pause to recall a few results in order to make our paper self contained as possible. Clearly, in [2, Proposition 1], if P is a ring property (i.e., P is inherited by every subring and every homomorphic image), then P is called an h-property. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then P is called an H-property. A ring property P

By lemma 2.4, we get

$$(y+1)^r[x,(y+1)^m] = y^r[x,y^m] \tag{2.3}$$

for all $x, y \in R$ and some fixed integers $r \geq 0, s \geq 0$ and $m > 1$.

Equation (2.3) is a polynomial identity and we observe that $x = e_{11}+e_{12}$ and $y = e_{12}$ in $(GF(p))_2, p$ a prime, fail to satisfy the equality in (2.3). Hence by Lemma 2.2, $C(R) \subseteq N(R)$. □

Result 2.2. *Let R be a ring satisfying (P_1). Then $C(R) \subseteq N(R)$.*

Proof. If R satisfies the property (P_1), then by using a similar technique of replacing y by $y+1$, and together with Lemma 2.4, we find that R satisfies the polynomial identity

$$[x,(y+1)^m](y+1)^r = [x,y^m]y^r$$

for all $x, y \in R$, and some fixed integers $r \geq 0, m > 1$. But $x = e_{11}$ and $y = e_{12}$ in $(GF(p))_2, p$ a prime, fail to satisfy the above equality. Hence Lemma 2.2 yields that $C(R) \subseteq N(R)$. □

Result 2.3. *Let R be a ring with 1 satisfying the property (P). If R has the property $Q(m)$, then $N(R) \subseteq Z(R)$.*

Proof. Let R satisfy the property (P) and let a be an arbitrary element in $N(R)$. Then there exists an integer $t \geq 1$ such that

$$a^k \in Z(R), \quad \text{for all integers } k > t, \quad t \text{ minimal.} \tag{2.4}$$

If $t = 1$, then $a \in Z(R)$, that is $N(R) \subseteq Z(R)$. Suppose that $t > 1$. Replacing y by a^{t-1} in (P), we get

$$a^{r(t-1)}[x, a^{m(t-1)}]x^s = x^p[x^n, a^{t-1}]x^q.$$

In view of (2.4), and the fact that $m(t-1) \geq t$, for integer $m > 1$, we get

$$x^p[x^n, a^{t-1}]x^q = 0, \qquad \text{for all } x \text{ in } R. \tag{2.5}$$

Replacing y by $1 + a^{t-1}$ in (P), we get

$$(1+a^{t-1})^r[x,(1+a^{t-1})^m]x^s = x^p[x^n, a^{t-1}]x^q.$$

The last expression, together with Lemma 2.2, gives

$$(1+a^{t-1})^r[x,(1+a^{t-1})^m] = 0$$

for all x in R. Since $(1 + a^{t-1})$ is invertible, the last equation implies that

$$[x,(1+a^{t-1})^m] = 0 \quad \text{for all } x \text{ in } R. \tag{2.6}$$

Combining (2.4) and (2.6), we get

$$0 = [x,(1+a^{t-1})^m] = [x, 1+ma^{t-1}] = m[x, a^{t-1}].$$

$x \in R$. A ring R is called s-unital if it is both left as well as right s-unital, that is, $x \in xR \cap Rx$ for each $x \in R$.

There are several results in the existing literature [1, 5, 6, 9, 10] concerning the commutativity of rings satisfying special cases of the following ring properties:

(P) For each x in R, there exist integers $n = n(x) \geq 0$, $p = p(x) \geq 0$ $s = s(x) \geq 0$ and $q = q(x) \geq 0$ such that

$$y^r[x, y^m]x^s = x^p[x^n, y]x^q \tag{1.1}$$

for all y in R, with fixed positive integers r and m.

(P_1) For each x in R, there exist integers $n = n(x) \geq 0$, $p = p(x) \geq 0$ $s = s(x) \geq 0$ and $q = q(x) \geq 0$ such that

$$x^s[x, y^m]y^r = x^p[x^n, y]x^q \tag{1.2}$$

for all y in R, with fixed positive integers r and m.

To establish the commutativity of a ring R satisfying anyone of the above properties, we need the following condition.

$Q(m)$: For all x, y in R, $m[x, y] = 0$ implies $[x, y] = 0$, where m is some integer.

Recently, the Khan [6] has shown that a ring with unity 1 is commutative if, for every x, y in R, R satisfies one of the polynomial identities $x^t[x^n, y]y^r = \pm y^s[y^m, x]$ and $x^t[x, y^m]y^s = \pm[y^m, x]y^s$, where $m > 1, n \geq 1$ and r, s, t are fixed non-negative integers with the property $Q(n)$. In most of the cases, the exponents in the above conditions have been considered "global". Up to now, some papers [5, 7, 8] on commutativity of rings have published. The results when the exponents in the underlying conditions are "local", that is, they are dependent on the ring's elements for their values. The aim of the present paper is to investigate commutativity of certain rings satisfying (P) or (P_1). In Section 2, we shall prove the commutativity of rings satisfying the above properties. However, in Section 3, we extend these results to the wider class of rings that are called one-sided s-unital.

2. Commutativity of Rings With Unity 1

Theorem 2.1. *Let R be a ring with unity 1 satisfying the property (P). If R has the property $Q(m)$, then R is commutative.*