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# On the ELzaki Transform and Ordinary Differential Equation with Variable Coefficients

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**Abstract**

The ELzaki transform, whose fundamental properties are presented in this paper, is little known and not widely used .Here The ELzaki transform used to solve ordinary differential equation with variable coefficients without resorting to anew frequency domain

**Keyword:** Elzaki transform- differential equations

## Introduction

A new integral transform, called the ELzaki transform defined for functions of exponential order, is proclaimed. We Consider function in the set A, defined by

$$A=\left\{ f(t):\ \exists M,\ k_1\ and\ k_2>0: \left|f(t)\right|<Me^{|t|k_j}\ ,\ if\quad t\in(-1)^j\times[0,\infty) \right\} \quad (1)$$

For a given function in the set $A$, the constant $M$ must be finite, while $k_1$ and $k_2$ may be infinite, the variable $v$ in the ELzaki transform is used to factor the variable $t$ in the argument of the function $f$ .specifically, for $f(t)$ in $A$ . The ELzaki transform is defined by:

$$\mathrm{E}\left[f(t)\right]=v\int_0^{\infty} f(t)\ e^{-t/v}\,dt=T(v)\quad ,\ v\in(-k_1,k_2) \qquad (2)$$

The next theorem very useful in study of differential equations having non-constants coefficient.

**Theorem I**

If ELzaki transform of the function $f(t)$ given by $\mathrm{E}\left[f(t)\right]=T(v)$, then:

$$(i)\ \mathrm{E}\left[tf'(t)\right]=v^2\frac{d}{dv}\left[\frac{T(v)}{v}-v\ f(0)\right]-v\left[\frac{T(v)}{v}-v\ f(0)\right]$$

$$(ii)\ \ \mathrm{E}\left[t^2f'(t)\right]=v^4\frac{d^2}{dv^2}\left[\frac{T(v)}{v}-v\ f(0)\right]$$

$$(iii)\mathrm{E}\left[tf''(t)\right]=v^2\frac{d}{dv}\left[\frac{T(v)}{v^2}-f(0)-vf'(0)\right]$$

$$-v\left[\frac{T(v)}{v^2}-f(0)-vf'(0)\right]$$

$$(iv)\quad (iii)\mathrm{E}\left[t^2f''(t)\right]=v^4\frac{d^2}{dv^2}\left[\frac{T(v)}{v^2}-f(0)-vf'(0)\right]$$

**Proof**

To Prove (i) we use the following formula

$$\mathrm{E}\left[t\,f(t)\right]=v^2\frac{d}{dv}T(v)-vT(v)$$

We have

$$\frac{d}{dv}T(v)=T'(v)=\frac{d}{dv}\int_0^\infty v\,e^{-t/v}f(t)\,dt$$

$$=\int_0^\infty\frac{\partial}{\partial v}v\,e^{-t/v}f(t)\,dt=\int_0^\infty\frac{1}{v}e^{-t/v}\left(t\,f(t)\right)dt$$

$$+\int_0^\infty e^{-t/v}f(t)\,dt=\frac{1}{v^2}\mathrm{E}\left(t\,f(t)\right)+\frac{1}{v}\mathrm{E}\left(f(t)\right)$$

Then we have

$$\mathrm{E}\left(tf(t)\right)=v^2\frac{d}{dv}T(v)-vT(v)$$

Now we put $f(t)=f'(t)$ we have

$$\mathrm{E}\left(tf'(t)\right)=v^2\frac{d}{dv}\left[\mathrm{E}\left(f'(t)\right)\right]-v\ \mathrm{E}\left[f'(t)\right]$$

$$=v^2\frac{d}{dv}\left[\frac{T(v)}{v}-vf(0)\right]-v\left[\frac{T(v)}{v}-vf(0)\right]$$

The proof of $(ii)$,$(iii)$ and $(iv)$ are similar to the Proof of $(i)$.

Now we apply the above theorem to find ELzaki transform for some differential equations:

**Example I**

Solve the differential equation:

$$y'' + ty' - y = 0 \quad , \qquad y(0) = 0 \quad , y'(0) = 1, \; t > 0 \tag{3}$$

By using ELzaki transform into equation (3) and the theorem I, we have

$$\frac{\mathrm{E}(y)}{v^2} - y(0) - vy'(0) + v^2 \frac{d}{dv}\left[\frac{\mathrm{E}(v)}{v} - vy(0)\right] - v\left[\frac{\mathrm{E}(y)}{v} - vy(0)\right] - E(y) = 0$$

Using the initial Conditions we get

$$\mathrm{E}'(y) + \left(\frac{1}{v^3} - \frac{3}{v}\right)\mathrm{E}(y) = 1$$

This is a linear differential equation for unknown function $\mathrm{E}$, have the Solution in the form $\mathrm{E}(y) = v^3 + C\,v^3\,e^{\frac{1}{2v^2}}$ and $C = 0$, then:

$$\mathrm{E}(y) = v^3.$$

By using the inverse ELzaki transform we obtain the Solution in the form of $y = t$

**Example II**

Consider the non constant coefficient differential equation in the form of

$$t\,y'' + (1 - 2t)\,y' - 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 2$$

By using ELzaki transform and apply the initial condition we have

$$\frac{\mathrm{E}'(y)}{\mathrm{E}(y)} = \frac{2}{v} + \frac{2}{1-2v}$$

Then we obtain the solution $\mathrm{E}(y) = \dfrac{Cv^2}{1-2v}$ where $C$ is constant

$$\mathrm{E}(y) = T(v) \Rightarrow T(0) = 1$$

Then the constant $C = 1$ and $\mathrm{E}(y) = \dfrac{v^2}{1-2v}$

By taking inverse ELzaki transform we have:

$$y = e^{2t}$$

**Example III**

Consider the following equation:

$$t^2 y' + 2ty = \sinh t \quad , \; y(0) = \frac{1}{2} \tag{4}$$

**Solution**

By taking ELzaki transform of equation (4) we have,

$$v^4 \frac{d^2}{dv^2}\left[\frac{\bar{y}}{v} - v y(0)\right] + 2v^2 \frac{d}{dv}\bar{y} - 2v\bar{y} = \frac{v^3}{1-v^2}$$

Where that $\bar{y} \equiv \mathrm{E}\left[y(t)\right]$

We can write the last equation in the form

$$v^3 \bar{y}'' = \frac{v^3}{1-v^2} \quad or \quad \bar{y}'' = \sum_{i=0}^{\infty} v^{2i}$$

The solution of this equation is

$$\bar{y} = c_1 + c_2 v + \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)} \tag{5}$$

Substituting the condition into $Eq(5)$ we get:

$$\bar{y} = \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)} \tag{6}$$

By taking inverse ELzaki transform to equation (6) we obtain the solution as follow,

$$y = \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i+2)!} = \frac{1}{2!} + \frac{t^2}{4!} + \frac{t^4}{6!} + \frac{t^6}{8!} + \ldots..$$

**Example IV**

Let us consider the differential equation

$$t^2 y'' + 4t\, y' + 2y = 12t^2 \tag{7}$$

With the initial conditions:

$$y(0) = y'(0) = 0 \tag{8}$$

Now we apply ELzaki transform to equation (7) we obtain,

$$v^4 \frac{d^2}{dv^2}\left[\frac{\bar{y}}{v^2} - y(0) - vy'(0)\right] + 4v^2 \frac{d}{dv}\left[\frac{\bar{y}}{v} - y(0)\right]$$

$$-4\,v\left[\frac{\bar{y}}{v} - vy(0)\right] + 2\,\bar{y} = 24\ v^4$$

By simplifying above equation, we have $\bar{y}'' = 24\,v^2$

The solution of this equation can be written in the form.

$$\bar{y} = 2\ v^4 + C_1\,v + C_2 \tag{9}$$

By substituting the initial condition (8) into equation (9) we get,

$$\bar{y} = 2\ v^4 \tag{10}$$

By using inverse ELzaki transform for equation (10) we obtain the solution of equation (7)

$$y = t^2$$

## Conclusion

Application of the ELzaki transform to Solution of ordinary differential equation with variable Coefficients has been demonstrated.

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