LINEAR SPACES

1. Linear Spaces

Linear spaces play a big role in functional analysis and its applications. The definition will involve a general field $\mathbb{F}$ where $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{F}$ are called *scalars*.

Definition 1.1:[Linear(Vector) Space]

A linear space over $\mathbb{F}$ is a none-empty set X of objects called *vectors* along with two operation

$$+ : X \times X \longrightarrow X \quad (x,y) \mapsto x+y, \forall x,y \in X \qquad \text{addition of vectors}$$
$$\cdot : \mathbb{F} \times X \longrightarrow X \quad (\alpha,x) \mapsto \alpha x, \forall \alpha \in \mathbb{F}, x \in X \qquad \text{scalar multiplication of vectors}$$

satisfying the following conditions

(1) $x+y=y+x \quad \forall x,y \in X$ (Commutative)
(2) $(x+y)+z=x+(y+z) \quad \forall x,y,z \in X$ (Associative)
(3) there exist a unique *zero vector* $\mathbf{0} \in X$ such that $x+\mathbf{0}=x \quad \forall x \in X$.
(4) $\forall x \in X$, there exist $-x \in X$ such that $x+(-x)=\mathbf{0}$.
(5) $\alpha(x+y)=\alpha x+\alpha y, \quad \forall x,y \in X$ and $\forall \alpha \in \mathbb{F}$.
(6) $(\alpha+\beta)x=\alpha x+\beta y \quad \forall x \in X$ and $\forall \alpha,\beta \in \mathbb{F}$.
(7) $\alpha(\beta x)=\alpha\beta x \quad \forall x \in X$ and $\forall \alpha,\beta \in \mathbb{F}$.
(8) $1x=x \quad \forall x \in X$.

Example 1: Let $X=\{\mathbf{0}\}$ .Then X is a linear space and is called the *zero space* over $\mathbb{F}$.

Example 2: $\mathbb{R}^n, \quad n \geq 1$. The Euclidean space. $\mathbb{R}^n=\{x \mid x=(x_1,x_2,\cdots,x_n), x_1,x_2,\cdots,x_n \in \mathbb{R}\}$ where

$$x+y=(x_1,x_2,\cdots,x_n)+(y_1,y_2,\cdots,y_n)=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n) \text{ and}$$
$$\alpha x=(\alpha x_1,\alpha x_2,\cdots,\alpha x_n), \quad \alpha \in \mathbb{R}.$$

Example 3: $\mathbb{C}^n, \quad n \geq 1$. The Euclidean space. $\mathbb{C}^n=\{x \mid x=(x_1,x_2,\cdots,x_n), x_1,x_2,\cdots,x_n \in \mathbb{C}\}$ where

$$x+y=(x_1,x_2,\cdots,x_n)+(y_1,y_2,\cdots,y_n)=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n) \text{ and}$$
$$\alpha x=(\alpha x_1,\alpha x_2,\cdots,\alpha x_n), \quad \alpha \in \mathbb{C}.$$

Example 4: l_p. Let p be a real number such that $1 \leq p < \infty$. l_p is the space of all sequence $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{F}$ such that $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty \quad (x = \{x_n\}_{n=1}^{\infty}$ converges).

$l_p = \{x = \{x_n\}_{n=1}^{\infty} \mid \sum\limits_{n=1}^{\infty} |x_n|^p < \infty, \quad x_n, \in \mathbb{F}, \forall n \in \mathbb{N}\}$ where

$$\begin{aligned} x+y =& (x_1, x_2, \cdots, x_n, \cdots) + (y_1, y_2, \cdots, y_n, \cdots) = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n, \cdots) \text{ and} \\ \alpha x =& (\alpha x_1, \alpha x_2, \cdots, \alpha x_n, \cdots), \quad \alpha \in \mathbb{F}. \end{aligned}$$

We will prove that if $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ are elements in l_p, then $x+y \in l_p$. Now, we have

$$\begin{aligned} |x_n+y_n|^p &\leq 2^p \max\{|x_n|^p, |y_n|^p\} \\ &\leq 2^p(|x_n|^p + |y_n|^p). \\ \text{Hence } \sum_{n=1}^{\infty} |x_n+y_n|^p &\leq 2^p \sum_{n=1}^{\infty} |x_n|^p + 2^p \sum_{n=1}^{\infty} |y_n|^p \\ &< \infty. \end{aligned}$$

Thus $x+y \in l_p$. Now, it is easy to verify that l_p is a linear space.

Example 5: Each of the following is a linear space with operations defined in Example 6.

$$\begin{aligned} x+y =& (x_1, x_2, \cdots, x_n, \cdots) + (y_1, y_2, \cdots, y_n, \cdots) = (x_1+y_1, x_2+y_2, \cdots, x_n+y_n, \cdots) \text{ and} \\ \alpha x =& (\alpha x_1, \alpha x_2, \cdots, \alpha x_n, \cdots) \quad \alpha \in \mathbb{F}. \end{aligned}$$

(1) The set of all sequences in $\mathbb{F}$, $\quad \omega = \{x = \{x_n\}_{n=1}^{\infty} \mid x_n, \in \mathbb{F}\}$

(2) The set of all convergent sequences in $\mathbb{F}$, $\quad c = \{x = \{x_n\}_{n=1}^{\infty} \mid \lim\limits_{n\to\infty} x_n \in \mathbb{F}, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$

(3) The set of all sequences in $\mathbb{F}$ converging to 0, $\quad c_0 = \{x = \{x_n\}_{n=1}^{\infty} \mid \lim\limits_{n\to\infty} x_n = 0, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$

(4) The set of all bounded sequences in $\mathbb{F}$, $\quad l_\infty = \{x = \{x_n\}_{n=1}^{\infty} \mid \sup\limits_{n\in\mathbb{N}} |x_n| < \infty, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$

Example 6: $C([a,b])$. Let a, b be two real numbers such that $a < b$. $C([a,b])$ is the space of all continuous real-valued functions f over $[a,b]$.

$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f$ is continuous on $[a,b]\}$ where

$$\begin{aligned} (f+g)(x) =& f(x) + g(x) \quad \forall f, g \in C([a,b]), x \in [a,b] \text{ and} \\ (\alpha f)(x) =& \alpha f(x), \quad \alpha \in \mathbb{R}. \end{aligned}$$

Example 7: $L_p([a,b])$. Let a,b be two real numbers such that $a<b$. $L_p([a,b])$ is the space of all Lebesgue measurable functions f such that $\int_a^b |f|^p < \infty$.

$L_p([a,b]) = \{f : [a,b] \to \mathbb{R} \mid \int_a^b |f|^p < \infty\}$ where

$$(f+g)(x) = f(x)+g(x) \quad \forall f,g \in L_p([a,b]), x \in [a,b] \text{ and}$$
$$(\alpha f)(x) = \alpha f(x), \quad \alpha \in \mathbb{R}.$$

Lemma 1: Let X be a linear space over $\mathbb{F}$ then:

(1) $0x = \mathbf{0}, \quad \forall x \in X.$

(2) $\alpha\mathbf{0} = \mathbf{0}, \quad \forall \alpha \in \mathbb{F}.$

(3) $(-1)x = -x, \quad \forall x \in X.$

(4) $\alpha x = \mathbf{0} \Rightarrow x = \mathbf{0} \quad \text{or } \alpha = 0.$

Proof: We will prove (1) and (3) and the reader should do (2) and (4).

1.

$$0x = (0+0)x$$
$$0x = 0x + 0x \qquad \text{Using property 6}$$
$$0x + (-0x) = 0x + 0x + (-0x)$$
$$\mathbf{0} = 0x \qquad \text{Using property 4}$$

3.

$$(-1)x + x = (-1+1)x \qquad \text{Using property 7}$$
$$(-1)x + x = 0x \qquad \text{Using part 1 of this Lemma}$$
$$(-1)x + x = \mathbf{0} \qquad \text{Using part 1 of this Lemma}$$
$$(-1)x + x + (-x) = \mathbf{0} + (-x)$$
$$(-1)x = -x \qquad \text{Using property 4}$$

Subspaces, Linearly Independent and Basis. we will study the geometry of the linear space.

Definition 1.2:[Subspace]

A non-empty subset Y of a linear space X over $\mathbb{F}$ is called a linear subspace if

$$\alpha x + \beta y \in Y, \forall x,y \in Y, \text{ and } \forall \alpha,\beta \in \mathbb{F}.$$

Example 8: Let $n \in \mathbb{N}, n \geq 1$, then $\mathbb{R}^{n-1} = \{(x_1, x_2, \cdots, x_{n-1}, 0) \mid x_1, x_2, \cdots, x_{n-1} \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^n$. Also $P[a,b]$, the set of all polynomials over $[a,b]$ is a subspace of $C[a,b]$.

Definition 1.3:[Linearly independent and basis]

a. A finite set of vectors $\{x_1, x_2, \cdots, x_n\}$ in a linear space X is called *linearly independent* if for any scalars $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}$ we have

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = \mathbf{0} \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

b. A subset Y of X is *linearly independent* if every non-empty finite subset of Y is linearly independent and Y is called linearly dependent if it not linearly independent.

c. A subset B of X is called a basis for X if

1. B is a linearly independent set.
2. $X =$ Span $B = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \mid \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}, x_1, x_2, \cdots, x_n \in B\}$

Example 9: Let $n \in \mathbb{N}, n \geq 1$. Let

$$\begin{aligned} e_1 =& (1,0,0,\cdots,0), \\ e_2 =& (0,1,0,\cdots,0), \\ &\vdots \\ e_n =& (0,0,0,\cdots,1). \end{aligned}$$

Then the set $B = \{e_1, e_2, \cdots, e_n\}$ is a linearly independent set that span $\mathbb{R}^n$. Hence B is a basis for $\mathbb{R}^n$.

Example 10: Let $n \in \mathbb{N}, n \geq 1$. Let

$$\begin{aligned} f_1(x) =& x, \ \forall x \in [a,b] \\ f_2(x) =& x^2, \ \forall x \in [a,b] \\ &\vdots \\ f_n(x) =& x^n, \ \forall x \in [a,b] \\ &\vdots \end{aligned}$$

Then the set $B = \{f_1, f_2, \cdots, f_n, \cdots\}$ is a linearly independent set that span $P[a,b]$. Hence B is a basis for $P[a,b]$.

Theorem 1.1: []

(1) Every linear space $X \neq \{\mathbf{0}\}$ has a basis.

(2) Let X be a finite dimensional linear space. Then, all bases for X have the same number of elements.

Definition 1.4:[Dimension of a Linear Space]

(1) A linear space X is called *finite dimensional* if it has a finite basis and we defined the dimension of X, denoted by, $\dim X =$ the number of element of the basis .

(2) A linear space X is called *infinite dimensional* if it has no a finite basis and we defined the dimension of X, $\dim X = \infty$.

Direct Sums. Let X be a linear space, and let $x \in X$. Let Y, Z be two subspace of X, and let $\alpha \in \mathbb{F}$. Define the following

$$Y + Z = \{y + z : y \in Y, z \in Z\} \qquad \text{The sum of two subspaces}$$

$$Y - Z = \{y - z : y \in Y, z \in Z\} \qquad \text{Algebraic Difference}$$

$$x + Y = \{x + y : y \in Y\} \qquad \text{Translate of } Y \text{ by } x.$$

$$x - Y = \{x - y : y \in Y\}$$

$$\alpha Y = \{\alpha y : y \in Y\}$$

Definition 1.5:[Direct Sum]

A linear space X is called the *direct sum* of two subspaces Y and Z of X, denoted by $X = Y \oplus Z$, if the following two conditions hold:

1. $X = Y + Z$
2. $Y \cap Z = \{\mathbf{0}\}$

The subspace $Y(Z)$ is called an algebraic complement of $Z(Y)$ in X.

Quotient Spaces. Let Y be a subspace of linear space X. The cost of an element $x \in X$ with respect to Y, $x + Y = \{x + y : y \in Y\}$. Note that any two cosets are either disjoint or identical. Also note that the distinct cosets form a partition of X. Let $X/Y = \{x + Y : x \in X\}$ and define the algebraic operation by

$$(x + Y) + (z + Y) = (x + z) + Y \qquad \forall x, z \in X.$$

$$\alpha(x + Y) = \alpha x + Y \quad \forall x \in X, \forall \alpha \in \mathbb{F}.$$

Under the algebraic operation defined above X/Y is a linear space and it is called the *quotient space* of X by Y. The dimension of the space X/Y is called codimension of Y and is denoted by codim$Y$ so codim$Y = \dim(X/Y)$. The function $q : X \to X/Y$ defined by $q(x) = x + Y, \forall x \in X$ is called *the quotient mapping(the canonical mapping)*

EXERCISES FOR SECTION 1

1. Let $f_n(x) = x^n, \forall x \in [a,b], n \in \mathbb{N}$. Show that the set $\{f_1, f_2, \cdots, f_n\}$ is a linearly independent set in the linear space $C[a,b]$.
2. Let $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \right\}$.
   Show that $M_2(\mathbb{R})$ is a linear space over $\mathbb{R}$. Give examples of subspaces of $M_2(\mathbb{R})$.
3. Prove parts (2) and (4) of Lemma 1.
4. Let $X = \mathbb{C}^3$ and let $Y = \{(0,x,0) \mid x \in \mathbb{C}\}$. Find $X/Y, X/\{0\}, X/X$.
5. Let X_1 and X_2 be any two linear spaces over $\mathbb{F}$. Show that $X = X_1 \times X_2$ withe algebraic operation by

$$(x_1,x_2) + (y_1,y_2) = (x_1 + y_1, x_2 + y_2), \quad x_1, y_1 \in X_1, x_2, y_2 \in X_2 \text{ and}$$

$$\alpha(x_1,x_2) = (\alpha x_1, \alpha x_2), \quad \alpha \in \mathbb{F}.$$

   is a linear space over $\mathbb{F}$.

