THE DERIVATIVE

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Definition 0.1. Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say $f'(c)$ is the ***derivative of f at*** c if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x-a| < \delta \Rightarrow |\dfrac{f(x)-f(c)}{x-c} - f'(c)| < \varepsilon$. In other word, the derivative of f at c is given by the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists.

Example 0.1. Let $f(x) = a,\ a \in \mathbb{R}$. Prove that $f'(c) = 0\ \forall\ c \in \mathbb{R}$.

Solution:

We have $f(x) = a$, and $f(c) = a$. Then

$$\begin{aligned} f'(c) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \to c} \frac{a - a}{x - c} \\ & \lim_{x \to c} \frac{0}{x - c}. \\ &= \lim_{x \to c} (0) \\ &= 0. \end{aligned}$$

Hence $f'(c) = 0$.

Example 0.2. Let $f(x) = x^2$. Prove that $f'(c) = 2c, \ \forall \ c \in \mathbb{R}$.

Solution:

We have $f(x) = x^2$, and $f(c) = c^2$. Then

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x\to c} \frac{\cancel{(x - c)}(x + c)}{\cancel{x - c}}. \\ &= \lim_{x\to c}(x + c) \\ &= c + c \\ &= 2c. \end{aligned}$$

Hence $f'(c) = 2c$.

Example 0.3. Let $f(x) = x^n, \ n \in \mathbb{N}$. Prove that $f'(c) = nc^{n-1}, \ \forall \ c \in \mathbb{R}$.

Solution:

We have $f(x) = x^n$, and $f(c) = c^n$, and note that $x^n - c^n = (x - c)\left(\sum_{k=1}^{n} x^{n-k}c^{k-1}\right)$. Then

$$\begin{aligned} f'(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x\to c} \frac{x^n - c^n}{x - c} \\ &= \lim_{x\to c} \frac{\cancel{(x - c)}\left(\sum_{k=1}^{n} x^{n-k}c^{k-1}\right)}{\cancel{x - c}}. \\ &= \lim_{x\to c}\left(\sum_{k=1}^{n} x^{n-k}c^{k-1}\right) \\ &= \sum_{k=1}^{n} c^{n-k}c^{k-1} \\ &= \sum_{k=1}^{n} c^{n-1} \\ &= nc^{n-1}. \end{aligned}$$

Hence $f'(c) = nc^{n-1}$.

Theorem 0.1. *Let $f : (a, b) \to \mathbb{R}$ and let $c \in (a, b)$. Then f is differentiable at c with derivative $f'(c)$ if and only if for every sequence $\{x_n\} \subseteq (a, b)$ such that $\lim_{n\to\infty} x_n = c$ and $x_n \neq c \ \forall \ n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = f'(c).$$

Proof. We have proved a similar theorem in the Limits section. □

Theorem 0.2. *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

Proof. For all $x \in I, \ x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c}\right)(x - c).$$

Now,

$$\begin{aligned}
\lim_{x\to c}(f(x)) - f(c) = \lim_{x\to c}(f(x) - f(c)) &= \lim_{x\to c}\left[\left(\frac{f(x) - f(c)}{x - c}\right)(x - c)\right] \\
&= \lim_{x\to c}\left(\frac{f(x) - f(c)}{x - c}\right) . \lim_{x\to c}(x - c) \\
&= f'(c).0 \\
&= 0.
\end{aligned}$$

Therefore $\lim_{x\to c} f(x) = f(c)$ so f is continuous at c.

□

Example 0.4. Let $f(x) = |x|$. Prove that f is not differentiable at 0.

Solution:

For each $n \in \mathbb{N}$, let $x_n = \frac{1}{n}$ and $y_n = \frac{-1}{n}$. Then $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$. Now,

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{f(\frac{1}{n}) - f(0)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n} - 0}{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{f(\frac{-1}{n}) - f(0)}{\frac{-1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n} - 0}{\frac{-1}{n}} = \lim_{n\to\infty} (-1) = -1.$$

Hence $\lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exists.

Theorem 0.3. *Let $f, g : (a, b) \to \mathbb{R}$ are differentiable at $c \in (a, b)$. Then*

(a) $(f \pm g)'(c) = f'(c) \pm g'(c)$.

(b) $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

(c) $\left(\dfrac{f}{g}\right)'(c) = \dfrac{g(c)f'(c) - g'(c)f(c)}{[g(c)]^2}$ *if* $g(c) \neq 0$.

Proof. We will prove parts **(a)** and **(c)**.

$$\begin{aligned}
\textbf{(a)}(f+g)'(c) &= \lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x - c} \\
&= \lim_{x\to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\
&= \lim_{x\to c} \frac{(f(x) - f(c)) + (g(x) - g(c))}{x - c}. \\
&= \lim_{x\to c} \left[\frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}\right] \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{x - c} + \lim_{x\to c} \frac{g(x) - g(c)}{x - c} \\
&= f'(c) + g'(c).
\end{aligned}$$

(c)
$$\left(\frac{f}{g}\right)'(c) = \lim_{x\to c}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(c)}{x-c}$$
$$= \lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c}$$
$$= \lim_{x\to c}\frac{\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)}}{x-c}.$$
$$= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}$$
$$= \lim_{x\to c}\frac{g(c)[f(x)-f(c)]-f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}$$
$$= \lim_{x\to c}\left[\frac{g(c)[f(x)-f(c)]}{g(x)g(c)(x-c)}-\frac{f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}\right]$$
$$= \lim_{x\to c}\left[\frac{g(c)[f(x)-f(c)]}{g(x)g(c)(x-c)}\right]-\lim_{x\to c}\left[\frac{f(c)[g(x)-g(c)]}{g(x)g(c)(x-c)}\right]$$
$$= \left[\lim_{x\to c}\left[\frac{g(c)}{g(c)g(x)}\right]\lim_{x\to c}\left[\frac{f(x)-f(c)}{(x-c)}\right]\right]-\left[\lim_{x\to c}\left[\frac{f(c)}{g(c)g(x)}\right]\lim_{x\to c}\left[\frac{g(x)-g(c)}{(x-c)}\right]\right]$$
$$= \left[\frac{g(c)}{[g(c)]^2}.f'(c)\right]-\left[\frac{f(c)}{[g(c)]^2}.g'(c)\right]$$
$$= \left[\frac{g(c)f'(c)}{[g(c)]^2}-\frac{f(c)g'(c)}{[g(c)]^2}\right]$$
$$= \frac{g(c)f'(c)-f(c)g'(c)}{[g(c)]^2}.$$

□

Definition 0.2. Let $I\subseteq\mathbb{R}$ be an interval, let $f: I\to\mathbb{R}$, and let $c\in I$.

(i) We say f has a ***relative maximum at*** c if there exists $\delta>0$ such that $f(x)\le f(c)$, $\forall\, x\in(c-\delta,c+\delta)$.

(ii) We say f has a ***relative minimum at*** c if there exists $\delta>0$ such that $f(c)\le f(x)$, $\forall\, x\in(c-\delta,c+\delta)$.

(iii) We say f has a ***relative extremum at*** c if f has either a relative maximum or relative minimum at c.

Theorem 0.4. *Let $f:(a,b)\to\mathbb{R}$ and let $c\in(a,b)$. If f has a relative extremum at c and $f'(c)$ exists, then $f'(c)=0$.*

Proof. Suppose f has a relative maximum at c. [We will prove that $f'(c)=0$.] Then there exists $\delta>0$ such that $f(x)\le f(c)$, $\forall\, x\in(c-\delta,c+\delta)$. Now, if $c-\delta<x<c$, then $f(x)\le f(c)\Rightarrow f(x)-f(c)\le 0$ and $x-c<0$. Hence $\frac{f(x)-f(c)}{x-c}\ge 0$. Thus $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\ge 0$. Therefore $f'(c)\ge 0$ (1).
Also, if $c<x<c+\delta$, then $f(x)\le f(c)\Rightarrow f(x)-f(c)\le 0$ and $x-c>0$. Hence $\frac{f(x)-f(c)}{x-c}\le 0$. Thus $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\le 0$. Therefore $f'(c)\le 0$ (2). By (1) and (2) we get $f'(c)=0$. □

Theorem 0.5. *[* **Rolle's Theorem:** *] Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, then there exist $c, d \in [a, b]$ such that $f(c) = \sup\{f(x) : x \in [a, b]\}$ and $f(d) = \inf\{f(x) : x \in [a, b]\}$. Now, if $f(c) = f(d)$, then $f(x)$ is constant and $f'(x) = 0, \ \forall \ x \in [a, b]$. If $f(c) \neq f(d)$, since $f(a) = f(b)$, then at least $f(c) \neq f(a)$ or $f(d) \neq f(a)$. Suppose $f(c) \neq f(a)$, then $c \in (a, b)$ and f has a relative maximum at c. Hence $f'(c) = 0$. □

Theorem 0.6. *[* **Mean Value Theorem:** *] Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.*

Proof. Let $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - f(a) - \left[\dfrac{f(b) - f(a)}{b - a}\right](x - a)$. Then g is continuous on $[a, b]$ and is differentiable on (a, b). Now, $g(a) = f(a) - f(a) - \left[\dfrac{f(b) - f(a)}{b - a}\right](a - a) = 0$ and $g(b) = f(a) - f(b) - \left[\dfrac{f(b) - f(a)}{b - a}\right](b - a) = 0$. Hence g satisfies the hypotheses of Rolle's Theorem. Then there exists $c \in (a, b)$ such that $g'(c) = 0$. Thus $0 = g'(c) = f'(c) - \left[\dfrac{f(b) - f(a)}{b - a}\right]$.
Therefore $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Example 0.5. Prove that $e^x \geq x + 1, \ \forall \ x \in \mathbb{R}$.

Solution:

Let $f(t) = e^t$ then f is continuous and differentiable on $\mathbb{R}$. Now, if $x \in \mathbb{R}$, then on the interval $[0, x]$ or $[x, 0]$ f satisfies the M.V.T. Hence there exists c such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$. Thus $e^c = \dfrac{e^x - e^0}{x}$. Hence $e^x - 1 = e^c x$. Since $e^c > 1$, then $e^x - 1 = e^c x > 1.x$. Thus $e^x - 1 > x$. Therefore $e^x > x + 1$.

Example 0.6. Prove that $|\sin x| \leq |x|$.

Solution:

Let $f(t) = \sin t$ then f is continuous and differentiable on $\mathbb{R}$. Now, if $x \in \mathbb{R}$, then on the interval $[0, x]$ or $[x, 0]$ f satisfies the M.V.T. Hence there exists c such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$. Thus $\cos c = \dfrac{\sin x - \sin 0}{x}$. Hence $\sin x - \sin 0 = x \cos c$. Since $|\cos c| \leq 1$, then $|\sin x| = |\cos c||x| \leq 1|x|$. Therefore $|\sin x| \leq |x|$.

