Partial Fractions

Suppose $f(x)=\dfrac{P(x)}{Q(x)}$ is a rational function; that is, $P(x)$ and $Q(x)$ are polynomial functions. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then by long division, $f(x)=\dfrac{P(x)}{Q(x)}=S(x)+\dfrac{R(x)}{Q(x)}$ where $\dfrac{R(x)}{Q(x)}$ is a proper rational fraction; that is, the degree of $R(x)$ is less than the degree of $Q(x)$. A theorem in advanced algebra states that every proper rational function can be expressed as a sum

$$\frac{R(x)}{Q(x)}=F_1(x)+F_2(x)+\cdots+F_n(x)$$

where $F_1(x),F_2(x),...,F_n(x)$ are rational functions of the form

$$\frac{A}{(ax+b)^k} \text{ or } \frac{Ax+B}{(ax^2+bx+c)^k}$$

in which the denominators are factors of $Q(x)$. The sum is called the *partial fraction decomposition* of $\dfrac{R(x)}{Q(x)}$. The first step is finding the form of the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$ is to factor $Q(x)$ completely into linear and irreducible quadratic factors, and then collect all repeated factors so that $Q(x)$ is expressed as a product of *distinct* factors of the form

$$(ax+b)^m \text{ and } (ax^2+bx+c)^m.$$

From these factors we can determine the form of the partial fraction decomposition using the following two rules:

***Linear Factor Rule*:** For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \ldots, A_m$ are constants to be determined.

***Quadratic Factor Rule*:** For each factor of the form $(ax^2+bx+c)^m$, the partial fraction decomposition contains the following sum of m partial fractions:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ are constants to be determined.

I. Integrating Improper Rational Functions

Example 1: Express $\dfrac{2x+3}{x-1}$ in the form $A+\dfrac{B}{x-1}$.

Solution:

$$\frac{2x+3}{x-1}=A+\frac{B}{x-1}=\frac{A(x-1)+B}{x-1}=\frac{Ax-A+B}{x-1}$$

Comparing coefficients with the original fraction gives, then $A=2$, $-A+B=3$, then $B=5$.

$$\frac{2x+3}{x-1}=A+\frac{B}{x-1}=2+\frac{5}{x-1}$$

Example 2: Find $\int \frac{x^3+1}{x-1}dx$.

Solution: By synthetic division, then $\frac{x^3+1}{x-1}=x^2+x+1+\frac{2}{x-1}$. Thus,

$$\int \frac{x^3+1}{x-1}dx=\int (x^2+x+1)dx+2\int \frac{1}{x-1}dx=\frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|\ x-1\ |+C.$$

Example 3: Find $\int \frac{x^3}{x^2-x-2}dx$.

Solution: By long division , $\frac{x^3}{x^2-x-2}=x+1+\frac{3x+2}{x^2-x-2}$. Thus,

$$\int \frac{x^3}{x^2-x-2}dx=\int (x+1)dx+\int \frac{3x+2}{x^2-x-2}dx .$$

$$\begin{array}{r|l} & x+1 \\ \hline x^2-x-2 & x^3+0x^2+0x+0 \\ & x^3-x^2-2x \\ \hline & x^2+2x \\ & x^2-x-2 \\ \hline & 3x+2 \end{array}$$

By partial fraction decomposition,

$$\int \frac{3x+2}{x^2-x-2}dx=\frac{8}{3}\ln|x-2|+\frac{1}{3}\ln|x+1|.$$

$$\int \frac{x^3}{x^2-x-2}dx=\frac{1}{2}x^2+x+\frac{8}{3}\ln|x-2|+\frac{1}{3}\ln|x+1|+C.$$

Example 4: Express $\frac{5x+2}{x^2-x-2}$ in partial fractions.

Solution: Factories the denominator and separate the factors to give two fractions:

$$\frac{5x+2}{(x-2)(x+1)}=\frac{A}{(x-2)}+\frac{B}{(x+1)}$$

Express again as a single fraction:

$$\frac{5x+2}{(x-2)(x+1)}=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}=\frac{(A+B)x+(A-2B)}{(x-2)(x+1)}$$

Compare coefficients: $A+B=5$ & $A-2B=2$, then $B=1$ & $A=4$.

$$\frac{5x+2}{(x-2)(x+1)}=\frac{4}{x-2}+\frac{1}{x+1}$$

II. Integrating Proper Rational Functions

Example 5: Find $\int \frac{3x-17}{x^2-2x-3}dx$.

Solution: $x^2-2x-3=(x-3)(x+1)\Rightarrow$ using the Linear Factor Rule, we get

$$\frac{3x-17}{x^2-2x-3}=\frac{3x-17}{(x-3)(x+1)}=\frac{A}{x-3}+\frac{B}{x+1}$$

$$\Leftrightarrow\ 3x-17=A(x+1)+B(x-3)$$

If we let $x=3$, then $-8=4A \Rightarrow A=-2$; if we let $x=-1$, then $-20=-4B \Rightarrow B=5$. Thus,

$$\int \frac{3x-17}{x^2-2x-3}\,dx = \int \frac{3x-17}{(x-3)(x+1)}dx = -2\int \frac{1}{x-3}dx + 5\int \frac{1}{x+1}dx$$

$$= -2\ln|x-3| + 5\ln|x+1| + C.$$

Example 6: Find $\int \frac{3x-4}{x^2-4x+4}\,dx$.

Solution: $x^2-4x-4=(x-2)(x-2)=(x-2)^2 \Rightarrow$ by the Linear Factor Rule, we get

$$\frac{3x-4}{x^2-4x+4} = \frac{3x-4}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$$

$$\Leftrightarrow \quad 3x-4 = A(x-2) + B$$

If we let $x = 2$, then $2=B$; if we let $x = 3$, then $5=A+B=A+2 \Rightarrow A=3$. Thus,

$$\int \frac{3x-4}{x^2-4x+4}\,dx = 3\int \frac{1}{x-2}dx + 2\int \frac{1}{(x-2)^2}dx = 3\ln|x-2| - \frac{2}{x-2} + C.$$

Example 7: Find $\int \frac{4x^2+x-2}{x^3-x^2}dx$.

Solution: $x^3-x^2=x^2(x-1) \Rightarrow$ using the Linear Factor Rule with both x^2 (multiplicity of linear factors) and $x-1$, we get

$$\frac{4x^2+x-2}{x^3-x^2} = \frac{4x^2+x-2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$\Leftrightarrow \quad 4x^2+x-2 = Ax(x-1) + B(x-1) + Cx^2$$

If we let $x = 0$, then $-2=B(-1) \Rightarrow B=2$; if we let $x = 1$, then $C=3$; and, if we let $x = 2$, then $16=2A+B+4C=2A+2+12 \Rightarrow 2=2A \Rightarrow A=1$. Thus,

$$\int \frac{4x^2+x-2}{x^3-x^2}dx = \int \frac{1}{x}\,dx + 2\int \frac{1}{x^2}dx + 3\int \frac{1}{x-1}dx = \ln|x| - \frac{2}{x} + 3\ln|x-1| + C.$$

Example 8: Find $\int \frac{x-1}{x^2+1}dx$.

Solution:

$$\int \frac{x-1}{x^2+1}dx = \int \frac{x}{x^2+1}dx - \int \frac{1}{x^2+1}dx = \frac{1}{2}\int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx =$$

$$= \frac{1}{2}\ln|x^2+1| - \arctan x + C.$$

Note: This problem doesn't really illustrate the Quadratic Factor Rule, but it does illustrate how to split a fraction with a linear numerator and a quadratic denominator into the sum or difference of two fractions with the same quadratic denominator.

Example 9: Find $\int \frac{7x^2+x+2}{(x-1)(x^2+1)}dx$.

Solution: Using both the Linear Factor Rule and the Quadratic Factor Rule, we get

$$\frac{7x^2+x+2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Leftrightarrow \quad 7x^2+x+2 = A(x^2+1) + (Bx+C)(x-1)$$

If we let $x=1$, then $10=2A \Rightarrow A=5$; if we let $x=0$, then $2=A-C=5-C \Rightarrow C=3$; and if we let $x=2$, then 32 = $5A+2B+C=25+2B+3 \Rightarrow 2B=4 \Rightarrow B=2$. Thus,

$$\int \frac{7x^2+x+2}{(x-1)(x^2+1)}dx = \int \frac{5}{x-1}dx + \int \frac{2x+3}{x^2+1}dx = 5\int \frac{1}{x-1}dx + \int \frac{2x}{x^2+1}dx + 3\int \frac{1}{x^2+1}dx$$

$$= 5\ln\left| x-1 \right| + \ln\left| x^2+1 \right| + 3\arctan x + C.$$

Example 10: Find $\int \frac{5x^2+11}{(x^2+1)(x^2+4)}dx$.

Solution:

$$\frac{5x^2+11}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$\Leftrightarrow\ 5x^2+11 = (Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

First method: Compare coefficients

$$5x^2+11 = (Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

x^3 coeff.	$0=A+C \Rightarrow A=-C$	(1)
x^2 coeff.	$5=B+D \Rightarrow D=5-B$	(2)
x coeff.	$0=4A+C$	(3)
const.. coeff.	$11=4B+D$	(4)

From (1) and (3) we get $C=0$ & $A=0$. From (2) and (4) we get $11=4B+5-B \Rightarrow B=2$ & $D=3$.

Second method:

If we let $x=0$, then $11=4B+D$; if we let $x=1$, then $16=5A+5B+2C+2D$; if we let $x=-1$,then $16=-5A+5B-2C+2D$; and if $x=2$, then $31=16A+8B+10C+5D$. If we add the equations $16=5A+5B+2C+2D$ and $16=-5A+5B-2C+2D$, we get $32=10B+4D \Rightarrow$ combining this equation with $11=4B+D$, we get $B=2$ and $D=3$. If we subtract the equations $16=5A+5B+2C+2D$ and $16=-5A+5B-2C+2D$, we get $0=10A+4C \Rightarrow 2C=-5A \Rightarrow$ combining this equation with $31=16A+8B+10C+5D$, $B=2$, and $D=3$, we get 31 = $16A+16-25A+15 \Rightarrow A=0 \Rightarrow C=0$. Thus,

$$\int \frac{5x^2+11}{(x^2+1)(x^2+4)}dx = 2\int \frac{1}{x^2+1}dx + 3\int \frac{1}{x^2+4}dx = 2\arctan x + \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C.$$

Example 11: Find $\int \frac{1}{x^4+x^2}dx$.

Solution:

$$\frac{1}{x^4+x^2} = \frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\Leftrightarrow\quad 1 = Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2$$

If we let $x=0$, then $1=B$. If we let $x=1$, then $1=2A+2B+C+D \Rightarrow 2A+C+D=-1$. If we let $x=-1$, then $1=-2A+2B-C+D \Rightarrow -2A-C+D=-1 \Rightarrow$ adding this equation to the equation $2A+C+D=-1$, we get $D=-1$. If we let $x=2$, then $1=10A+5B+8C+4D \Rightarrow 0=10A+8C$. Taking the equations $0=10A+8C$ and $2A+C=0$, we obviously get that $A=0$ and $C=0$. Thus,

$$\int \frac{1}{x^4+x^2}dx = \int \frac{1}{x^2}dx - \int \frac{1}{x^2+1}dx = \frac{-1}{x} - \arctan x + C.$$

Preview

Guidelines for Partial Fraction Decomposition of $\dfrac{f(x)}{g(x)}$

1. If the degree of $f(x)$ is not lower than the degree of $g(x)$, use long division to obtain the proper form.

2. Express $g(x)$ as a product of linear factors $ax+b$ or irreducible quadratic ax^2+bx+c, and collect repeated factors so that $g(x)$ is a product of different factors of the form $(ax+b)^n$ or $(ax^2+bx+c)^n$ for a nonnegative integer *n*.

3. Apply the following rules.

Case I. <u>Distinct Linear Factors</u>
To each linear factor $ax+b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{A}{ax+b}$, where *A* is a constant to be determined.

Case II. <u>Repeated Linear Factors</u>
To each linear factor $ax+b$ occurring *n* times in the denominator of a proper rational fraction, there corresponds a sum of *n* partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$$

where the *A*'s are constants to be determined.

Case III. <u>Distinct Quadratic Factors</u>
To each irreducible quadratic factor ax^2+bx+c occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where *A* and *B* are constants to be determined.

Case IV. <u>Repeated Quadratic Factors</u>
To each irreducible quadratic factor ax^2+bx+c occurring *n* times in the denominator of a proper rational fraction, there corresponds a sum of *n* partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where the *A*'s and *B*'s are constants to be determined.

Problems:

1. Express in the form $A + \frac{B}{x+C}$ when A, B and C are constants:

a) $\frac{x+2}{x+1}$ b) $\frac{x-1}{x-2}$ c) $\frac{2x+6}{x+2}$

d) $\frac{2x-7}{x-3}$ e) $\frac{4x+8}{x-3}$ f) $\frac{9x-5}{3x-5}$

2. Express in partial fractions:

a) $\frac{2}{(x-1)(x+1)}$ b) $\frac{5}{(x+2)(x-2)}$ c) $\frac{4}{(2x-1)(x-2)}$

d) $\frac{x+3}{x^2+x}$ e) $\frac{x+2}{x^2-x}$ f) $\frac{5x+1}{(x+1)(3x+1)}$

3. Express in partial fractions:

a) $\frac{x^2-3}{(x+1)(x-1)}$ b) $\frac{x^3-2x^2}{(x-3)(x+1)}$ c) $\frac{x^3+x^2}{(x+2)(x-1)}$

4-13. Find the following integrals

4. $\int \frac{-x^2+3x+4}{x(x+2)^2}\,dx$ 5. $\int \frac{4x+2}{(x-1)(x^2+1)}\,dx$

6. $\int \frac{x-6}{x^2-2x}\,dx$ 7. $\int \frac{1}{u^2-a^2}\,dx$

8. $\int \frac{3x^2+x+1}{(x-1)(x^2+4)}\,dx$ 9. $\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx$

10. $\int \frac{e^x}{e^{2x}+3e^x+2}\,dx$ 11. $\int \frac{1}{x\sqrt{x+1}}\,dx$

12. $\int \frac{x^3+1}{x^2+1}\,dx$ 13. $\int \frac{6x^2-7x+6}{x^2(x-2)}\,dx$

