Series of Real Numbers

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Definition 5.1: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

We call $a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$ an infinite series , and a_n is the *nth term* of the series .

For $n \geq 1$, let

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2 = S_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3 = S_2 + a_3, \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n = S_{n-1} + a_n. \end{aligned}$$

We call the sequence $\{S_n\}_{n=1}^{\infty}$ the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$. We say that $\sum_{n=1}^{\infty} a_n$ *converges* to a real number S, and we write $\sum_{n=1}^{\infty} a_n = S$ if the sequence $\{S_n\}_{n=1}^{\infty}$ converge to S $[\lim_{n\to\infty} S_n = S]$. We say that $\sum_{n=1}^{\infty} a_n$ *diverges* if the sequence $\{S_n\}_{n=1}^{\infty}$ diverges.

Example 5.1: Consider the series $\sum_{n=1}^{\infty} (-1)^n$. Discuses the convergence of the series.

Solution:

Let us find the sequence of partial sum. Note that $S_n = a_1 + a_2 + \cdots + a_n = S_{n-1} + a_n$

$$\begin{aligned} S_1 &= a_1 = -1, \\ S_2 &= a_1 + a_2 = S_1 + a_2 = -1 + 1 = 0, \\ S_3 &= a_1 + a_2 + a_3 = S_2 + a_3 = 0 - 1, \\ &\vdots \\ S_n &= \begin{cases} 0, & \text{if } n \text{ is even;} \\ -1, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence $\lim_{n\to\infty} S_n$ does not exist . Therefore $\sum_{n=1}^{\infty} (-1)^n$ diverges .

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Example 5.2: Consider the series $\sum_{n=1}^{\infty} 2^{-n}$. Discuses the convergence of the series.

Solution:

Let us find the sequence of partial sum.

$$S_1 = a_1 = \frac{1}{2},$$
$$S_2 = \frac{1}{2} + \frac{1}{2^2},$$
$$S_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3},$$
$$\vdots$$
$$S_n = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n}.$$
$$\text{Now, } 2S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}}.$$
$$\text{Hence, } 2S_n - S_n = \left[1 + \cancel{\frac{1}{2}} + \cancel{\frac{1}{2^2}} + \cdots + \cancel{\frac{1}{2^{n-2}}} + \cancel{\frac{1}{2^{n-1}}}\right] - \left[\cancel{\frac{1}{2}} + \cancel{\frac{1}{2^2}} + \cdots + \cancel{\frac{1}{2^{n-1}}} + \frac{1}{2^n}\right]$$
$$= 1 - \frac{1}{2^n}.$$
$$\text{Hence } S_n = 1 - \frac{1}{2^n}.$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty} [1 - \frac{1}{2^n}] = 1$. Therefore $\sum_{n=1}^{\infty} 2^{-n}$ converges and

$$\sum_{n=1}^{\infty} 2^{-n} = 1.$$

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Remark 5.1: If $\{a_n\}$ is an increasing sequence of positive real numbers, then

(i) $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$, if $\{a_n\}$ is bounded,

(ii) $\lim_{n\to\infty} a_n = \infty$, if $\{a_n\}$ is unbounded.

(iii) If $\{a_{n_k}\}$ is an unbounded subsequence of $\{a_n\}$, then $\{a_n\}$ is unbounded.

Lemma 5.1: [Harmonic Series]

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof: For each $n \geq 1$ we have $S_{n+1} \geq S_n$. Thus $\{S_n\}$ is an increasing sequence. We will try to find a subsequence

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of the sequence of partial sum that is unbounded.

$$S_{2^0} = S_1 = 1,$$
$$S_{2^1} = S_2 = 1 + \frac{1}{2},$$
$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = S_{2^1} + \left[\frac{1}{3} + \frac{1}{4}\right] > S_{2^1} + \left[\frac{1}{4} + \frac{1}{4}\right] = 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2},$$
$$S_{2^3} = S_8 = 1 + \frac{1}{2} + \cdots + \frac{1}{8} = S_{2^2} + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] > S_{2^2} + \left[\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right] > 1 + \frac{2}{2} + \frac{4}{8} = 1 + \frac{2}{2} + \frac{1}{2} = 1 + \frac{3}{2},$$
$$\vdots$$
$$S_{2^n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} > 1 + \frac{n}{2}.$$

Now, the subsequence $\{S_{2^n}\}$ of the sequence $\{S_n\}$ is unbounded. Hence $\{S_n\}$ is unbounded.

$$\text{Therefore } \lim_{n\to\infty} S_n = \infty, \text{ and the series } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.}$$

Lemma 5.2: [Geometric Series]

$$\text{The series } \sum_{n=1}^{\infty} a^n \text{ converges if and only if } |a| < 1 \text{ and } \sum_{n=1}^{\infty} a^n = \frac{a}{1-a}.$$

Proof: If $|a| > 1$, we have $\lim_{n\to\infty} a^n$ does not exist and if $|a| < 1 \Rightarrow \lim_{n\to\infty} a^n = \lim_{n\to\infty} a^{n+1} = 0$.

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} a^k = a + a^2 + \cdots + a^n, \\
aS_n &= a^2 + \cdots + a^n + a^{n+1}, \\
S_n - aS_n &= [a + a^2 + \cdots + a^n] - [a^2 + \cdots + a^n + a^{n+1}], \\
S_n(1-a) &= a - a^{n+1}. \text{ Hence } S_n = \frac{a - a^{n+1}}{1-a}, \qquad \text{if } a \neq 1. \\
\text{Now, } \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{a - a^{n+1}}{1-a} \\
&= \frac{a}{1-a} \lim_{n\to\infty} (1 - a^n) \\
&= \begin{cases} \dfrac{a}{1-a}, & \text{if and only if } |a| < 1; \\ \text{does not exist }, & \text{if and only if } |a| > 1. \end{cases} \\
\text{If } a = 1, \quad S_n &= \sum_{k=1}^{n} a^k = a + a^2 + \cdots + a^n = n \text{ and hence }, \lim_{n\to\infty} S_n = \lim_{n\to\infty} n = \infty.
\end{aligned}$$

Therefore $\sum_{n=1}^{\infty} a^n$ converges and $\sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$ if and only if $|a| < 1$.

Theorem 5.1: *[The Cauchy Criterion]*

The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$m > n > N \Rightarrow \left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.$$

Proof: Let $\{S_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$, and let $\epsilon > 0$ be given

$$\begin{aligned}
\text{The series } \sum_{n=1}^{\infty} x_n \text{ converges } &\Leftrightarrow \text{ the sequence } \{S_n\} \text{ converges} \\
&\Leftrightarrow \text{ the sequence } \{S_n\} \text{ is Cauchy sequence} \\
&\Leftrightarrow \exists N \in \mathbb{N} \ni \text{ if } m > n > N \Rightarrow |S_m - S_n| < \epsilon \\
&\Leftrightarrow \exists N \in \mathbb{N} \ni \text{ if } m > n > N \Rightarrow \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| < \epsilon \\
&\Leftrightarrow \exists N \in \mathbb{N} \ni \text{ if } m > n > N \Rightarrow \left| \cancel{\sum_{k=1}^{n} a_k} + \sum_{k=n+1}^{m} a_k - \cancel{\sum_{k=1}^{n} a_k} \right| < \epsilon \\
&\Leftrightarrow \exists N \in \mathbb{N} \ni \text{ if } m > n > N \Rightarrow \left| \sum_{k=n+1}^{m} a_k \right| < \epsilon.
\end{aligned}$$

■

Theorem 5.2: *[Divergence Test]*

If $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Let $\{S_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$. Note that $a_n = S_n - S_{n-1}$.

Now, since $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} S_n$ exist and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$. Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} [S_n - S_{n-1}] = 0$.

■

Remark 5.2: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note 5.1: Notice that $\lim_{n\to\infty} a_n = 0$ does not imply the convergence of $\sum_{n=1}^{\infty} a_n$. For example $\lim_{n\to\infty} \frac{1}{n} = 0$, but the $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series)is divergent.

Theorem 5.3: *[Integral Test]*

Let f be a positive, decreasing, and integrable on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges.

Proof: Let $\{S_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$. Note that $a_n = S_n - S_{n-1}$.

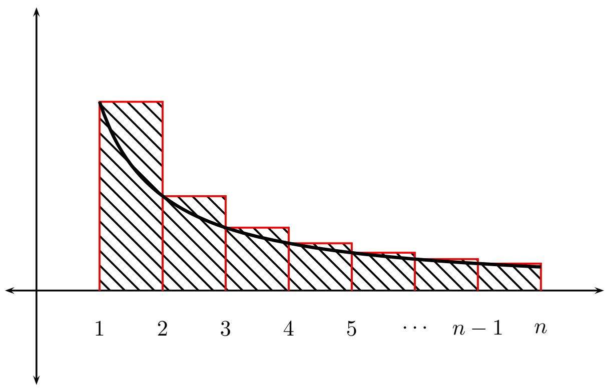

Figure 1: $U(f, P_n)$

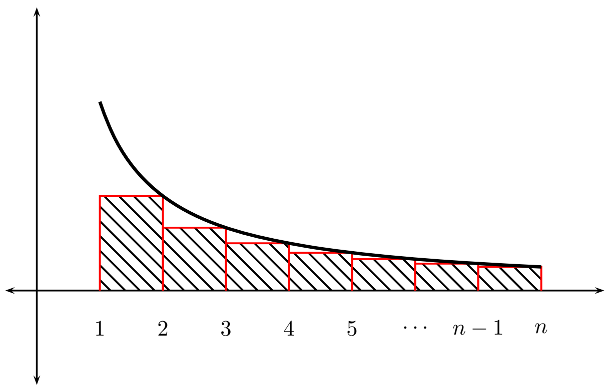

Figure 2: $L(f, P_n)$

Now, since $\displaystyle\int_1^\infty f(x)\,dx$ converges, then $\displaystyle\int_1^n f(x)\,dx$ exist $\forall\, n \geq 1$. Let $P_n = \{1 < 2 < 3 \cdots < n-1 < n\}$ be a partition of the interval $[1, n]$. Now, since f is integrable on $[1, n]$, then

$$L(f, P_n) \leq \int_1^n f(x)\,dx \leq U(f, P_n).$$

For the partition $P_n = \{1 < 2 < 3 \cdots < n-1 < n\}$, we have $M_k = f(k-1)$ and $m_k = f(k)$ because f is decreasing also we have $\triangle x_k = k - (k-1) = 1 \quad k = 2, 3 \cdots, n$.

Hence

$$\begin{aligned}
U(f, P_n) &= \sum_{k=2}^{n} M_k \triangle x_k \\
&= \sum_{k=2}^{n} f(k-1) \triangle x_k \\
&= \sum_{k=2}^{n} a_{k-1}(1) \\
&= a_1 + a_2 + \cdots + a_{n-1} \\
&= S_{n-1}.
\end{aligned}$$

Also,

$$\begin{aligned}
L(f, P_n) &= \sum_{k=2}^{n} m_k \triangle x_k \\
&= \sum_{k=2}^{n} f(k) \triangle x_k \\
&= \sum_{k=2}^{n} a_k(1) \\
&= a_2 + a_3 + \cdots + a_n \\
&= a_1 + a_2 + a_3 + \cdots + a_n - a_1 \\
&= S_n - a_1.
\end{aligned}$$

Hence we have

$$S_n - a_1 \le \int_1^n f(x)\,dx \le S_{n-1}.$$

Now, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1}$ exist and hence

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \int_1^{\infty} f(x)\,dx \le \lim_{n\to\infty} S_{n-1} < \infty.$$

Thus $\int_1^{\infty} f(x)\,dx$ converges.

Also, if $\int_1^{\infty} f(x)\,dx$ converges, then

$$\lim_{n\to\infty} S_n - a_1 \le \int_1^n f(x)\,dx \le \int_1^{\infty} f(x)\,dx < \infty.$$

Thus

$$\lim_{n\to\infty} S_n < \infty.$$

Thus the sequence $\{S_n\}$ converges and hence $\sum_{n=1}^{\infty} a_n$ converges

Theorem 5.4: *[p−Series Test]*

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof: If $p = 1$ we have the Harmonic series which is divergent. If $p \le 0$ we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^p} = \infty \ne 0$, hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by Divergence test. Now, if $p > 0$ consider the function $f(x) = \frac{1}{x^p}$ $x \ge 1$ then f is positive, decreasing, and integrable. Hence we can use the Integral test

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,dx &= \lim_{n\to\infty} \int_1^n x^{-p}\,dx \\
&= \lim_{n\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^n \\
&= \lim_{n\to\infty} \left[\frac{n^{1-p}-1}{1-p}\right] \quad \text{this limit has a finite value} \\
&\Leftrightarrow 1-p < 0 \\
&\Leftrightarrow p > 1
\end{aligned}$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$.

Theorem 5.5: *[Comparison Test]*

Suppose that $0 \le a_n \le b_n \,\forall\, n \ge 1$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.[if the bigger converges ⇒ the smaller converges]

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(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.[if the smaller diverges $\Rightarrow$ the bigger diverges]

Proof: Let $\{S_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$ and let $\{T_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} b_n$. Now, since $a_n \le b_n$, we have $S_n \le T_n$.

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} T_n$ exist and since $S_n \le T_n$ we have $\lim_{n\to\infty} S_n \le \lim_{n\to\infty} T_n < \infty$. Thus $\{T_n\}$ converges. Hence $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then since $\{S_n\}$ is an increasing positive sequence, we have $\lim_{n\to\infty} S_n = \infty$. Hence $\lim_{n\to\infty} T_n = \infty$. Thus $\{T_n\}$ is divergent and hence $\sum_{n=1}^{\infty} b_n$ diverges.

Example 5.3: Determine whether the given series converges or diverges

1.
$$\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$$

2.
$$\sum_{n=1}^{\infty} \frac{n}{n^2+2n+3}$$

Solution:

1.
$$\text{Since } 1 < \ln n \quad \forall\, n \ge 3 \Rightarrow \frac{1}{\sqrt{n}} < \frac{\ln n}{\sqrt{n}} \quad \forall\, n \ge 3 \text{ [if the smaller diverges} \Rightarrow \text{the bigger diverges]}$$

Now, since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ p-series with $p = \frac{1}{2} < 1$, diverges, then $\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges by Comparison Test.

2.
$$n^3+2n+3 > n^3 \Rightarrow \frac{1}{n^3+2n+3} < \frac{1}{n^3}$$
$$\Rightarrow \frac{n}{n^3+2n+3} < \frac{n}{n^3} = \frac{1}{n^2} \quad \text{[if the bigger converges} \Rightarrow \text{the smaller converges]}$$

Now, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series with $p = 2 > 1$, converges, then $\sum_{n=1}^{\infty} \frac{n}{n^2+2n+3}$ converges by Comparison Test.

Theorem 5.6: ***[Limit Comparison Test]***

Suppose that $a_n, b_n \ge 0\, \forall\, n \ge 1$, and $0 < \lim_{n\to\infty} \frac{a_n}{b_n} = c < \infty$. Then the two series $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n$ are both converge or diverge.

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Proof:

$$\text{Since } \lim_{n\to\infty}\frac{a_n}{b_n}=c \quad \therefore \quad \exists \quad N\in\mathbb{N} \ni$$
$$\text{if } n\geq N \Rightarrow \left|\frac{a_n}{b_n}-c\right|<\frac{c}{2}$$
$$\Rightarrow -\frac{c}{2}<\frac{a_n}{b_n}-c<\frac{c}{2}$$
$$\text{if } n\geq N \Rightarrow \frac{c}{2}<\frac{a_n}{b_n}<\frac{3c}{2}$$
$$\text{if } n\geq N \Rightarrow \frac{c}{2}b_n<a_n<\frac{3c}{2}b_n$$

Case I: if $\sum_{n=1}^{\infty} a_n$ converges. Since $\frac{c}{2}b_n < a_n$, then $\sum_{n=1}^{\infty} b_n$ converges by Comparison Test.

Case II: if $\sum_{n=1}^{\infty} a_n$ diverges. Since $a_n < \frac{3c}{2}b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges by Comparison Test.

Case III: if $\sum_{n=1}^{\infty} b_n$ converges. Since $a_n < \frac{3c}{2}b_n$, then $\sum_{n=1}^{\infty} a_n$ converges by Comparison Test.

Case IV: if $\sum_{n=1}^{\infty} b_n$ converges. Since $\frac{c}{2}b_n < a_n$, then $\sum_{n=1}^{\infty} a_n$ diverges by Comparison Test.

Example 5.4: Determine whether the given series converges or diverges

1. $$\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$$

2. $$\sum_{n=1}^{\infty}\frac{n}{n^3-n+1}$$

Solution:

1. Let $a_n=\sin\left(\frac{1}{n}\right)$ and $b_n=\frac{1}{n}$. Now, since $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1,$ and since

$\sum_{n=1}^{\infty}\frac{1}{n}$ p-series with $p=1,$ diverges , then $\sum_{n=1}^{\infty}\sin\left(\frac{1}{n}\right)$ diverges by Limit Comparison Test

2. Let $a_n=\frac{n}{n^3-n+1}$ and $b_n=\frac{n}{n^3}=\frac{1}{n^2}$. Now, since $\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{n}{n^3-n+1}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^3}{n^3-n+1}=1,$ and since

$\sum_{n=1}^{\infty}\frac{1}{n^2}$ p-series with $p=2>1,$ converges , then $\sum_{n=1}^{\infty}\frac{n}{n^3-n+1}$ converges by Limit Comparison Test

Definition 5.2: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ or $\sum_{n=1}^{\infty}(-1)^{n}a_n$ is called *alternating series.*

Theorem 5.7: ***[Alternating Series Test]***

The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if the following hold

1. $a_n > 0$
2. $a_n \geq a_{n+1}$
3. $\lim_{n\to\infty} a_n = 0$

Proof: Let $\{S_n\}$ be the sequence of partial sum of $\sum_{n=1}^{\infty} a_n$

$$S_2 = a_1 - a_2 \geq 0, \text{ because } a_1 \geq a_2$$
$$S_4 = a_1 - a_2 + a_3 - a_4 = (a_1 - a_2) + (a_3 - a_4) \geq 0, \text{ because } a_1 \geq a_2 \text{ and } a_3 \geq a_4$$
$$\vdots,$$
$$S_{2n} = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq 0,$$
$$\text{because } a_n \geq a_{n+1}$$

Hence $S_{2n} \geq 0$.

Also,

$$S_2 = a_1 - a_2 \leq a_1 \text{ because } a_1 \geq 0$$
$$S_4 = a_1 - (a_2 - a_3) - a_4 \leq a_1, \text{ because } a_2 - a_3 \geq 0 \text{ and } a_4 \geq 0$$
$$\vdots,$$
$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1, \text{ because } a_n \geq a_{n+1} \text{ and } a_n \geq 0$$

Hence $S_{2n} \leq a_1$.

Thus the subsequence $\{S_{2n}\}$ is an increasing bounded and hence it converges.

Thus $\lim_{n\to\infty} S_{2n}$ exist. Now, since $S_{2n+1} = S_{2n} + a_{2n+1}$ and $\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} a_n = 0$,

then we have $\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} S_{2n} + 0 = \lim_{n\to\infty} S_{2n}$.

Now, since the two subsequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge to the sam limit , then $\{S_n\}$ converges to that limit and hence the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Example 5.5: Determine whether the given series converges or diverges

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n}$$

Solution:

1.
$$\text{Let } a_n = \frac{1}{n}. \text{ Then } a_n = \frac{1}{n} > 0 \quad \forall\, n \in \mathbb{N}. \text{ Also, } a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n, \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by Alternating Series Test}$$

2.
$$\text{Let } a_n = \frac{1}{4^n} \text{ Then } a_n = \frac{1}{4^n} > 0 \quad \forall\, n \in \mathbb{N}. \text{ Also, } a_{n+1} = \frac{1}{4^{n+1}} < \frac{1}{4^n} = a_n, \text{ and } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4^n} = 0.$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} \text{ converges by Alternating Series Test}$$

Definition 5.3: Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers.

1. We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. We say that $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 5.6: Determine whether the given series converges absolutely or diverges conditionally.

1.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n}$$

Solution:

1.
$$\text{The series } \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges by Alternating Series Test.}$$

$$\text{Now } |a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}. \text{ Then } \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which is a Harmonic series diverges and hence ,}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges conditionally .}$$

2.

$$\text{The series } \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} \text{ converges by Alternating Series Test.}$$

$$\text{Now } |a_n| = \left|\frac{(-1)^n}{4^n}\right| = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n. \text{ Then } \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n \text{ which is a Geometric series with}$$

$$r = \frac{1}{4} < 1 \text{ converges and hence , } \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} \text{ converges absolutely .}$$

Theorem 5.8: ***[Ratio Test]***

Let $\sum_{n=1}^{\infty} a_n$ be a series and let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, there is no conclusion.

Proof: Note that $L \geq 0$ because $\frac{|a_{n+1}|}{|a_n|} > 0$.

1. Suppose that $0 \leq L < 1$. Let $L < x < 1$, then $x - L > 0$.

$$\text{Since } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \therefore \quad \exists\, N \in \mathbb{N} \ni$$

$$\text{if } n > N \Rightarrow \left|\frac{|a_{n+1}|}{|a_n|} - L\right| < x - L$$

$$\text{if } n > N \Rightarrow -(x - L) < \frac{|a_{n+1}|}{|a_n|} - L < x - L$$

$$\text{if } n > N \Rightarrow -x + L + L < \frac{|a_{n+1}|}{|a_n|} < x - L + L$$

$$\text{if } n > N \Rightarrow \frac{|a_{n+1}|}{|a_n|} < x = \frac{x^{n+1}}{x^n}$$

$$\text{if } n > N \Rightarrow \frac{|a_{n+1}|}{x^{n+1}} < \frac{|a_n|}{x^n}$$

$$\text{Hence if } n > N \Rightarrow \left\{\frac{|a_n|}{x^n}\right\} \text{ is a decreasing sequence and since } \frac{|a_n|}{x^n} > 0,$$

$$\text{then if } n > N \Rightarrow \left\{\frac{|a_n|}{x^n}\right\} \text{ is bounded } \exists\, M > 0 \ni \frac{|a_n|}{x^n} \leq M$$

$$\text{Hence if } n > N \Rightarrow |a_n| \leq x^n M \text{ and since the series } \sum_{n=N}^{\infty} Mx^n \text{ is a geometric series with } |x| < 1,$$

$$\text{then } \Rightarrow \sum_{n=N}^{\infty} Mx^n \text{ converges}$$

$$\text{Therefore } \Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ converges by Comparison Test}$$

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2. Suppose that $L > 1$. Then $L - 1 > 0$.

$$\begin{aligned}
\text{Since } \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= L \therefore \quad \exists\ N \in \mathbb{N} \ni \\
\text{if } n > N \Rightarrow & \left| \frac{|a_{n+1}|}{|a_n|} - L \right| < L - 1 \\
\text{if } n > N \Rightarrow & -(L-1) < \frac{|a_{n+1}|}{|a_n|} - L < L - 1 \\
\text{if } n > N \Rightarrow & -L + 1 + L < \frac{|a_{n+1}|}{|a_n|} < L + L - 1 \\
\text{if } n > N \Rightarrow & 1 < \frac{|a_{n+1}|}{|a_n|} \\
\text{if } n > N \Rightarrow & |a_n| < |a_{n+1}| \\
\text{Hence if } n > N \Rightarrow & \{|a_n|\} \text{ is a increasing sequence .} \\
\text{Then } & \lim_{n\to\infty} |a_n| \neq 0. \\
\text{Hence } & \lim_{n\to\infty} a_n \neq 0. \\
\text{Therefore } \Rightarrow & \sum_{n=1}^{\infty} a_n \text{ diverges by Divergence Test.}
\end{aligned}$$

Example 5.7: Determine whether the given series converges or diverges.

1.
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$$

2.
$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution:

1.
Let $a_n = \dfrac{(-2)^n}{n!}$. Then $a_{n+1} = \dfrac{(-2)^{n+1}}{(n+1)!} = \dfrac{-2.(-2)^n}{(n+1)n!}$. Hence , $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{-2.\cancel{(-2)^n}}{(n+1)\cancel{n!}}.\dfrac{\cancel{n!}}{\cancel{(-2)^n}}\right| = \dfrac{2}{n+1}$.

$$\text{Thus } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{(-2)^n}{n!} \text{ converges by Ratio Test}$$

Let $a_n = \dfrac{n!}{n^n}$. Then $a_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1}} = \dfrac{(n+1)n!}{(n+1)(n+1)^n}$. Hence , $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{\cancel{(n+1)}\cancel{n!}}{\cancel{(n+1)}(n+1)^n}.\dfrac{n^n}{\cancel{n!}}\right| = \dfrac{n^n}{(n+1)^n}$.

$$\text{Thus } \left|\frac{a_{n+1}}{a_n}\right| = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1-1}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^n.$$

$$\text{Thus } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n = e^{-1} < 1.$$

$$\text{Hence } \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges by Ratio Test}$$

Theorem 5.9: *[Root Test]*

Let $\sum_{n=1}^{\infty} a_n$ be a series and let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$ or $L = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, there is no conclusion.

Proof: Note that $L \geq 0$ because $\sqrt[n]{|a_n|} > 0$.

1. Suppose that $0 \leq L < 1$. Let $L < x < 1$, then $x - L > 0$.

$$\begin{aligned}
&\text{Since } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L \therefore \quad \exists\, N \in \mathbb{N} \ni \\
&\quad \text{if } n > N \Rightarrow \left| \sqrt[n]{|a_n|} - L \right| < x - L \\
&\quad \text{if } n > N \Rightarrow -(x - L) < \sqrt[n]{|a_n|} - L < x - L \\
&\quad \text{if } n > N \Rightarrow -x + L + L < \sqrt[n]{|a_n|} < x - L + L \\
&\quad \text{if } n > N \Rightarrow \sqrt[n]{|a_n|} < x
\end{aligned}$$

Hence if $n > N \Rightarrow |a_n| \leq x^n$ and since the series $\sum_{n=N}^{\infty} x^n$ is a geometric series with $|x| < 1$,

$$\text{then } \Rightarrow \sum_{n=N}^{\infty} M x^n \text{ converges}$$

$$\text{Therefore } \Rightarrow \sum_{n=1}^{\infty} |a_n| \text{ converges by Comparison Test}$$

2. Suppose that $L > 1$. Then $L - 1 > 0$.

$$\text{Since } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L \therefore \quad \exists\, N \in \mathbb{N} \ni$$
$$\text{if } n > N \Rightarrow \left| \sqrt[n]{|a_n|} - L \right| < L - 1$$
$$\text{if } n > N \Rightarrow -(L-1) < \sqrt[n]{|a_n|} - L < L - 1$$
$$\text{if } n > N \Rightarrow -L + 1 + L < \sqrt[n]{|a_n|} < L + L - 1$$
$$\text{if } n > N \Rightarrow 1 < \sqrt[n]{|a_n|}$$
$$\text{if } n > N \Rightarrow |a_n| > 1$$
$$\text{Then } \lim_{n\to\infty} |a_n| \geq 1.$$
$$\text{Hence } \lim_{n\to\infty} a_n \neq 0.$$
$$\text{Therefore } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges by Divergence Test.}$$

3. Suppose that $L = \infty$.

$$\text{Since } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty \therefore \quad \exists\, N \in \mathbb{N} \ni$$
$$\text{if } n > N \Rightarrow \sqrt[n]{|a_n|} > 1$$
$$\text{if } n > N \Rightarrow |a_n| > 1$$
$$\text{Then } \lim_{n\to\infty} |a_n| \geq 1.$$
$$\text{Hence } \lim_{n\to\infty} a_n \neq 0.$$
$$\text{Therefore } \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges by Divergence Test.}$$

Example 5.8: Determine whether the given series converges or diverges.

1.
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$$

2.
$$\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1} \right)^{2n}$$

Solution:

1.
$$\text{Let } a_n = \frac{(-2)^n}{n^n}. \text{ Then } |a_n| = \left| \frac{(-2)^n}{n^n} \right| = \left| \frac{-2}{n} \right|^n. \text{ Hence } \sqrt[n]{|a_n|} = \sqrt[n]{\left| \frac{-2}{n} \right|^n} = \left| \frac{-2}{n} \right| = \frac{2}{n}$$
$$\text{Thus } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1.$$
$$\text{Hence} \sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} \text{ converges by Root Test}$$

2.

Let $a_n = \left(\frac{n+1}{2n-1}\right)^{2n}$. Then $|a_n| = \left|\left(\frac{n+1}{2n-1}\right)^{2n}\right|$. Hence $\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{n+1}{2n-1}\right)^{2n}\right|^n} = \left(\frac{n+1}{2n-1}\right)^2$

$$\text{Thus } \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n+1}{2n-1}\right)^2 = \left(\lim_{n\to\infty} \frac{n+1}{2n-1}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1.$$

Hence $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n-1}\right)^{2n}$ converges by Root Test

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