Limits

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Definition 0.1: Let $E \subseteq \mathbb{R}$, and let a be a limit point of E. For a function $f : E \to \mathbb{R}$, a real number L is said to be a ***limit of*** f ***at*** a, we write $\lim\limits_{x\to a} f(x) = L$, if, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Note 0.1: The inequality $0 < |x-a|$ is equivalent to saying $x \neq a$.

Example 0.1: Prove that $\lim\limits_{x\to 2} x^2 = 4$.

Discussion: Given $\varepsilon > 0$, we want to find $\delta > 0$ such that if $0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$. Now,

$$\begin{aligned} |x^2-4| &= |(x-2)(x+2)| \\ &= |x-2||x+2| \\ &\le |x-2|(|x|+2). \end{aligned}$$

If we assume that $|x-2| < 10$, then $|x| - 2 < |x-2| < 10$. Hence $|x| - 2 < 10 \Rightarrow |x| < 12$.

$$\begin{aligned} \text{Now, } |x^2-4| &\le |x-2|(|x|+2) \\ &\le |x-2|(12+2) \\ &\le 14|x-2|. \end{aligned}$$

Now, if we assume $14|x-2| < \varepsilon \Rightarrow |x-2| < \dfrac{\varepsilon}{14}$.

Now, we have the following conditions on $|x-2|$: $|x-2| <10$ and $|x-2| < \dfrac{\varepsilon}{14}$. If we choose $\delta = \min\{10, \dfrac{\varepsilon}{14}\}$.

Proof: Let $\varepsilon > 0$ be given. Let $\delta = \min\{10, \dfrac{\varepsilon}{14}\}$.

$$\begin{aligned} \text{Now, if } 0 < |x-2| < \delta \Rightarrow |x^2-4| &\le 14|x-2| \\ &< 14\delta \\ &< 14.\frac{\varepsilon}{14} \\ &= \varepsilon. \end{aligned}$$

Thus, if $0 < |x-2| < \delta \Rightarrow |x^2-4| < \varepsilon$.

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Example 0.2: Prove that $\lim\limits_{x\to 1} \left(\dfrac{3x^2-1}{2x+1}\right) = \dfrac{2}{3}$. *Discussion:* Given $\varepsilon > 0$, we want to find $\delta > 0$ such that if

$0 < |x-1| < \delta \Rightarrow \left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right| < \varepsilon$. Now, the idea is to start with $\left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right|$ and try to make it less than or equal to $(number).|x-1|$.

$$\begin{aligned}
\left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right| &= \left|\frac{3(3x^2-1)-2(2x+1)}{3(2x+1)}\right| \\
&= \left|\frac{9x^2-3-4x-2}{3(2x+1)}\right| \\
&= \left|\frac{9x^2-4x-5}{3(2x+1)}\right| \\
&= \frac{|9x^2-4x-5|}{3|2x+1|} \\
&= \frac{|9x^2-9x+5x-5|}{3|2x+1|} \\
&= \frac{|9x(x-1)+5(x-1)|}{3|2x+1|} \\
&= \frac{|(9x+5)(x-1)|}{3|2x+1|} \\
&= \frac{|x-1||9x+5|}{3|2x+1|} \\
&\leq \frac{|x-1|(9|x|+5)}{3|2x+1|}
\end{aligned}$$

If we assume that $|x-1| < \frac{1}{2}$, then $|x|-1 < |x-1| < \frac{1}{2}$ and $\frac{-1}{2} < x-1 < \frac{1}{2}$. Hence $|x|-1 < \frac{1}{2} \Rightarrow |x| < \frac{3}{2}$, and

$$\begin{aligned}
\frac{-1}{2} < x-1 < \frac{1}{2} &\Rightarrow \frac{1}{2} < x < \frac{3}{2}. \\
&\Rightarrow 1 < 2x < 3 \\
&\Rightarrow 2 < 2x+1 < 4.
\end{aligned}$$

$$\begin{aligned}
\text{Thus } 2 < 2x+1 \leq |2x+1| &\Rightarrow \frac{1}{|2x+1|} < \frac{1}{2} \\
\text{Now, } \left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right| &\leq \frac{|x-1|(9|x|+5)}{3|2x+1|} \\
&\leq |x-1|\frac{1}{3|2x+1|}(9|x|+5) \\
&< |x-1|\frac{1}{3}\frac{1}{2}(9(\frac{3}{2})+5). \\
&= \frac{37}{12}|x-1|
\end{aligned}$$

Now, if we assume $\frac{37}{12}|x-1| < \varepsilon \Rightarrow |x-1| < \frac{12\varepsilon}{37}$.

Now, we have the following conditions on $|x-1| : |x-1| < \frac{1}{2}$ and $|x-1| < \frac{12\varepsilon}{37}$. If we choose $\delta = \min\{\frac{1}{2}, \frac{12\varepsilon}{37}\}$.

Proof: Let $\varepsilon > 0$ be given. Let $\delta = \min\{\frac{1}{2}, \frac{12\varepsilon}{37}\}$.

$$\begin{aligned} \text{Now, if } 0 < |x+1| < \delta \Rightarrow \left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right| &< |x-1|\frac{1}{3}\frac{1}{2}(9(\frac{3}{2})+5) \\ &= \frac{37}{12}|x-1| \\ &< \frac{37}{12}\delta \\ &< \frac{37}{12}.\frac{12\varepsilon}{37} \\ &= \varepsilon. \end{aligned}$$

$$\text{Thus, if } 0 < |x-1| < \delta \Rightarrow \left|\frac{3x^2-1}{2x+1} - \frac{2}{3}\right| < \varepsilon.$$

Example 0.3: Prove that $\lim_{x\to -1}(x^2+x+7) = 7$.

Discussion: Given $\varepsilon > 0$, we want to find $\delta > 0$ such that if $0 < |x-(-1)| = |x+1| < \delta \Rightarrow |x^2+x+7-7| < \varepsilon$. Now, the idea is to start with $|x^2+x+7-7|$ and try to make it less than or equal to $(number).|x+1|$.

$$\begin{aligned} |x^2+x+7-7| &= |x^2+x| \\ &= |x+1||x| \end{aligned}$$

If we assume that $|x+1| < 1$, then $|x|-1 < |x+1| < 1$. Hence $|x|-1 < 1 \Rightarrow |x| < 2$.

$$\begin{aligned} \text{Now, } |x^2+x+7-7| &\le |x+1||x| \\ &\le |x+1|2 \\ &\le 2|x+1|. \end{aligned}$$

Now, if we assume $2|x+1| < \varepsilon \Rightarrow |x+1| < \frac{\varepsilon}{2}$.

Now, we have the following conditions on $|x+1|$: $|x+1| <1$ and $|x+1| < \frac{\varepsilon}{2}$. If we choose $\delta = \min\{1, \frac{\varepsilon}{2}\}$.

Proof: Let $\varepsilon > 0$ be given. Let $\delta = \min\{1, \frac{\varepsilon}{2}\}$.

$$\begin{aligned} \text{Now, if } 0 < |x+1| < \delta \Rightarrow |x^2+x+7-7| &= |x+1||x| \\ &< 2|x+1| \\ &< 2\delta \\ &< 2.\frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

$$\text{Thus, if } 0 < |x+1| < \delta \Rightarrow |x^2+x+7-7| < \varepsilon.$$

Theorem 0.1: *[]*

Let $f : E \to \mathbb{R}$ and let a be a limit point of E. Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\} \subseteq E$ such that $\lim_{n\to\infty} x_n = a$ and $x_n \neq a \ \forall\ n \in \mathbb{N}$, then $\lim_{n\to\infty} f(x_n) = L$.

Proof: $(\Rightarrow)$ Suppose that $\lim_{x\to a} f(x) = L$. Let $\{x_n\} \subseteq E$ such that $\lim_{n\to\infty} x_n = a$ and $x_n \neq a\ \forall\ n \in \mathbb{N}$. We want to show that $\lim_{n\to\infty} f(x_n) = L$. Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, then there exist $\delta > 0$ such that $0 < |x-a| < \delta$, $\Rightarrow |f(x) - L| < \varepsilon$. Since, $\lim_{n\to\infty} x_n = a$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - a| < \delta$. Now, if $n > N \Rightarrow |x_n - a| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$. Hence , if $n > N \Rightarrow |f(x_n) - L| < \varepsilon$. Thus $\lim_{n\to\infty} f(x_n) = L$.

$(\Leftarrow)$ Suppose that for every sequence $\{x_n\} \subseteq E$ such that $\lim_{n\to\infty} x_n = a$ and $x_n \neq a\ \forall\ n \in \mathbb{N}$, then $\lim_{n\to\infty} f(x_n) = L$. Assume that $\lim_{x\to a} f(x) \neq L$. Then there exist $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exist $x \in E$ and $x \neq a$ with $0 < |x-a| < \delta$, but such that $|f(x) - L| \geq \varepsilon_0$. For all $n \in \mathbb{N}$, there exists $x_n \in E$ and $x_n \neq a$ with $0 < |x_n - a| < \frac{1}{n}$, but such that $|f(x_n) - L| \geq \varepsilon_0$. Hence we have a sequence $\{x_n\} \subseteq E$ such that $\lim_{n\to\infty} x_n = a$ and $x_n \neq a\ \forall\ n \in \mathbb{N}$, but the sequence $\{f(x_n)\}$ does not converges. Contradiction. Hence $\lim_{x\to a} f(x) = L$.

Note 0.2: The main use of this theorem is it usually used to prove the limit of some function does not exist at some limit point.

Example 0.4: Let $f(x) = \frac{1}{x}$. Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

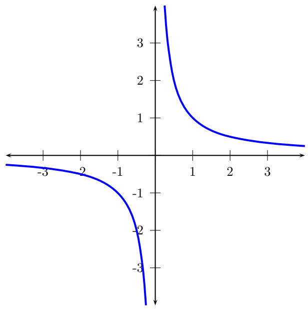

Figure 1: $y = \frac{1}{x}$

Solution: Let $\{x_n\} = \{\frac{1}{n}\} \subseteq \mathcal{D}(f)$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. Now $f(x_n) = f(\frac{1}{n}) = \frac{1}{\frac{1}{n}} = n$, and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = \infty$. Hence $\lim_{x\to 0} \frac{1}{x}$ DNE.

Example 0.5: Let $g(x) = \begin{cases} 2, & \text{if } x \geq 1; \\ 4, & \text{if } x < 1 \end{cases}$. Prove that $\lim_{x\to 1} g(x)$ does not exist.

Solution: Let $\{x_n\} = \{1+\frac{(-1)^n}{n}\} \subseteq \mathcal{D}(g)$ and $\lim_{n\to\infty}(1+\frac{(-1)^n}{n}) = 1$. Now, $g(x_n) = g\left(1 + \frac{(-1)^n}{n}\right) = \begin{cases} 2, & \text{if } n \text{ is even}; \\ 4, & \text{if } n \text{ is odd} \end{cases}$. Hence $\{g(x_n)\} = \{4, 2, 4, 2, \ldots\}$. Thus $\lim_{n\to\infty} g(x_n)$ DNE. Hence $\lim_{x\to 0} g(x)$ DNE.

Example 0.6: Let $g(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{Q}^c \end{cases}$. Let $a \in \mathbb{R} - \{0\}$, prove that $\lim_{x\to a} g(x)$ does not exist.

Solution: Let $a \in \mathbb{R} - \{0\}$. There exist two sequences $\{x_n\} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = a$ and $\{y_n\} \subseteq \mathbb{Q}^c$ such that $\lim_{n\to\infty} y_n = a$. Now, $g(x_n) = x_n$ and $g(y_n) = 0$. Hence $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} 0 = 0$. Since $a \neq 0$, then $\lim_{n\to\infty} g(y_n) = 0 \neq a = \lim_{n\to\infty} g(x_n)$. Thus $\lim_{x\to a} g(x)$ DNE for all $a \in \mathbb{R} - \{0\}$.

Definition 0.2: Let $E \subseteq \mathbb{R}$, and let a be a limit point of E. Let $f : E \to \mathbb{R}$. We say that f is ***bounded on an open interval about*** a , if there exists $\delta > 0$ and $M > 0$ such that $|f(x)| \leq M \; \forall \; x \in (a-\delta, x+\delta) \cap E$.

Theorem 0.2: []

Let $f : E \to \mathbb{R}$ and let a be a limit point of E. If $\lim_{x\to a} f(x)$ exists, then f is bounded on some open interval about a.

Proof: Let $L = \lim_{x\to a} f(x)$. Since $\lim_{x\to a} f(x)$ exists, then there exist $\delta > 0$ such that if $x \in E$ and $0 < |x-a| < \delta \; \Rightarrow |f(x)-L| < 7$. Hence if $x \in (a-\delta, a+\delta) \cap E$ and $x \neq a$, then $|f(x)|-|L| \leq |f(x)-L| < 7$. Thus if $x \in (a-\delta, a+\delta) \cap E$ and $x \neq a \Rightarrow |f(x)| < 7 + |L|$.

Now, if $a \notin E$, let $M = 7 + |L|$ and if $a \in E$ let $M = \sup\{7 + |L|, |f(a)|\}$.

Thus if $x \in (a-\delta, a+\delta) \cap E \Rightarrow |f(x)| \leq M$.

Lemma 0.1: If $\lim_{x\to a} f(x) = L$, then there exists $\delta > 0$ such that if $0 < |x-a| < \delta \Rightarrow \frac{|L|}{2} < |f(x)| < |L| + 1$.

Proof:

$$
\begin{aligned}
\text{Since } \lim_{x\to a} f(x) = L, \exists\, \delta_1 > 0 \text{ such that if } 0 < |x-a| < \delta_1 &\Rightarrow |f(x) - L| < 1 \\
&\Rightarrow |f(x)| - |L| \leq |f(x) - L| < 1 \\
&\Rightarrow |f(x)| < 1 + |L|
\end{aligned}
$$

$$\text{Thus } \exists\, \delta_1 > 0 \text{ such that if } 0 < |x-a| < \delta_1 \Rightarrow |f(x)| < 1 + |L|.$$

$$
\begin{aligned}
\text{Also since } \lim_{x\to a} f(x) = L, \exists\, \delta_2 > 0 \text{ such that if } 0 < |x-a| < \delta_2 &\Rightarrow |f(x) - L| < \frac{|L|}{2} \\
&\Rightarrow |L| - |f(x)| \leq |f(x) - L| < \frac{|L|}{2} \\
&\Rightarrow -|f(x)| < \frac{|L|}{2} - |L| \\
&\Rightarrow -|f(x)| < -\frac{|L|}{2} \\
&\Rightarrow \frac{|L|}{2} < |f(x)|
\end{aligned}
$$

$$\text{Thus } \exists\, \delta_2 > 0 \text{ such that if } 0 < |x-a| < \delta_2 \Rightarrow \frac{|L|}{2} < |f(x)|.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then if $0 < |x-a| < \delta \Rightarrow \frac{|L|}{2} < |f(x)| < 1 + |L|$.
