Continuity

Dr.Hamed Al-Sulami

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Definition 0.1: Let $f : E \to \mathbb{R}$, and let $a \in E$. We say f is ***continuous at*** a , if, for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon, a) > 0$ such that if $x \in E$ and $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

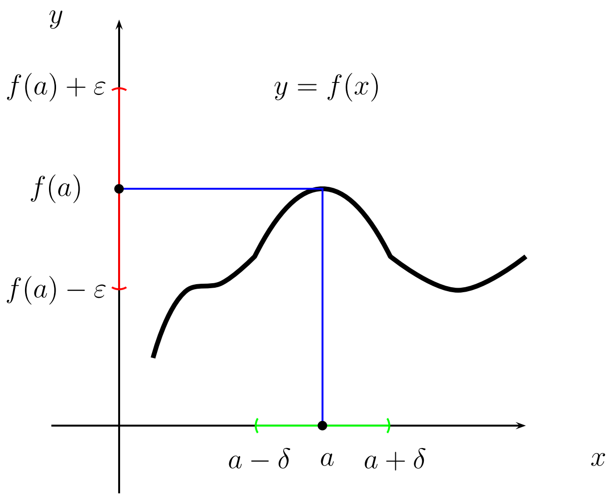

Figure 1:

Note 0.1:

- If f fails to be continuous at a, then we say f is **discontinuous at** a**.**
- This definition requires three things if f is continuous at a :
  - $f(a)$ is defined
  - $\lim_{x \to a} f(x)$ exists
  - $\lim_{x \to a} f(x) = f(a)$
- One can say f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

Example 0.1: Prove that $f(x) = x^2$ is continuous at $a \in \mathbb{R}$.

Discussion: Given $\epsilon > 0$, we want to find $\delta > 0$ such that if $|x-a| < \delta \Rightarrow |x^2 - a^2| < \epsilon$. Now,

$$\begin{aligned} |x^2 - a^2| &= |(x-a)(x+a)| \\ &= |x-a||x+a| \\ &\leq |x-a|(|x|+|a|). \end{aligned}$$

If we assume that $|x-a| < 1$, then $|x|-|a| < |x-a| < 1$. Hence $|x|-|a| < 1 \Rightarrow |x| < 1+|a|$.

$$\begin{aligned} \text{Now, } |x^2 - a^2| &\leq |x-a|(|x|+|a|) \\ &\leq |x-a|(1+|a|+|a|) \\ &\leq (1+2|a|)|x-a|. \end{aligned}$$

Now, if we assume $(1+2|a|)|x-a| < \varepsilon \Rightarrow |x-a| < \dfrac{\epsilon}{1+2|a|}$.

Now, we have the following conditions on $|x-a|$: $|x-a| <1$ and $|x-a| < \dfrac{\epsilon}{1+2|a|}$. If we choose $\delta = \min\{1, \dfrac{\epsilon}{1+2|a|}\}$.

Proof: Let $\epsilon > 0$ be given. Let $\delta = \min\{1, \dfrac{\epsilon}{1+2|a|}\}$.

$$\begin{aligned} \text{Now, if } |x-a| < \delta \Rightarrow |f(x) - f(a)| = |x^2 - a^2| &\leq (1+2|a|)|x-a| \\ &< (1+2|a|)\delta \\ &< (1+2|a|).\frac{\epsilon}{1+2|a|} \\ &= \epsilon. \end{aligned}$$

Thus, if $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Theorem 0.1: *[]*

Let $f : E \to \mathbb{R}$ and let $a \in E$. Then f is continuous at a if and only if for every sequence $\{x_n\} \subseteq E$ such that $\lim\limits_{n\to\infty} x_n = a$, then $\lim\limits_{n\to\infty} f(x_n) = f(a)$.

Proof: $(\Rightarrow)$ Suppose that f is continuous at a. Let $\{x_n\} \subseteq E$ such that $\lim\limits_{n\to\infty} x_n = a$. We want to show that $\lim\limits_{n\to\infty} f(x_n) = f(a)$. Let $\epsilon > 0$ be given.

Since f is continuous at a, then there exist $\delta > 0$ such that if $|x-a| < \delta, \Rightarrow |f(x) - f(a)| < \epsilon$. Since, $\lim\limits_{n\to\infty} x_n = a$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - a| < \delta$. Now, if $n > N \Rightarrow |x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \epsilon$. Hence , if $n > N \Rightarrow |f(x_n) - f(a)| < \epsilon$. Thus $\lim\limits_{n\to\infty} f(x_n) = f(a)$.

$(\Leftarrow)$ Suppose that for every sequence $\{x_n\} \subseteq E$ such that $\lim\limits_{n\to\infty} x_n = a$, then $\lim\limits_{n\to\infty} f(x_n) = f(a)$. Assume that f is discontinuous at a. Then there exist $\epsilon_0 > 0$ such that for all $\delta > 0$ there exist $x \in E$ with $|x-a| < \delta$, but such that $|f(x) - f(a)| \geq \epsilon_0$. For all $n \in \mathbb{N}$, there exists $x_n \in E$ with $|x_n - a| < \dfrac{1}{n}$, but such that $|f(x_n) - f(a)| \geq \epsilon_0$. Hence we have a sequence $\{x_n\} \subseteq E$ such that $\lim\limits_{n\to\infty} x_n = a$, but the sequence $\{f(x_n)\}$ does not converges. Contradiction. Hence f is continuous at a.

Example 0.2: Let $f(x) = \begin{cases} x+1, & \text{if } x \in \mathbb{Q}; \\ -2x+4, & \text{if } x \in \mathbb{Q}^c \end{cases}$. Discuses the continuity of f.

Solution: Let $a \in \mathbb{R} - \{1\}$.

Case I: If $a \in \mathbb{Q}$. There exists a sequence $\{y_n\} \subseteq \mathbb{Q}^c$ such that $\lim_{n\to\infty} y_n = a$. Now, $f(y_n) = -2y_n + 4$. Hence $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty}(-2y_n + 4) = -2a + 4 \neq a + 1 = f(a)$. Hence f is discontinuous at any $a \in \mathbb{Q} - \{1\}$.

Case II:

If $a \in \mathbb{Q}^c$. There exists a sequence $\{x_n\} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} x_n = a$. Now, $f(x_n) = x_n + 1$. Hence $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(x_n + 1) = a + 1 \neq -2a + 4 = f(a)$. Hence f is discontinuous at any $a \in \mathbb{Q}^c$. By the two cases we have f is discontinuous at any $a \in \mathbb{R} - \{1\}$. Now, to see that f is continuous at 1. Since $f(1) = 2$, then

$$|f(x) - f(1)| = \begin{cases} |x + 1 - 2|, & \text{if } x \in \mathbb{Q} \,; \\ |-2x + 4 - 2|, & \text{if } x \in \mathbb{Q}^c \,. \end{cases}$$

Hence

$$|f(x) - f(1)| = \begin{cases} |x - 1|, & \text{if } x \in \mathbb{Q}; \\ 2|x - 1|, & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Hence $|f(x) - f(1)| \leq \max\{|x - 1|, 2|x - 1|\} = 2|x - 1|$.

So, let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$. If $|x - 1| < \delta \Rightarrow |f(x) - f(1)| \leq 2|x - 1| < 2.\delta = 2\frac{\epsilon}{2} = \epsilon$.

Hence f is continuous at 1.

Definition 0.2: Let $f : E \to \mathbb{R}$, and let $C \subseteq E$. We say f is ***continuous on the set*** C , if f is continuous at every point of C.

Definition 0.3: A function $f : E \to \mathbb{R}$ is said to be ***bounded on*** E , if there exists a number $M > 0$ such that $|f(x)| \leq M, \ \forall\ x \in E$.

Note 0.2: A function f is not bounded on the set E if for all $M > 0$, there exists $x_M \in E$ such that $|f(x_M)| > M$.

Theorem 0.2: []

Let $[a, b]$ be closed bounded interval and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is bounded on I. Moreover, f assume its maximum and minimum values on $[a, b]$. [there exist $x_0, x_1 \in [a, b]$ such that $f(x_0) = \inf\{f(x) : x \in [a, b]\}$ and $f(x_1) = \sup\{f(x) : x \in [a, b]\}$.]

Proof: Suppose f is not bounded on $[a, b]$. Then for each $n \in \mathbb{N}$ there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. Now, we have a sequence $\{x_n\} \subseteq [a, b]$. Thus $\{x_n\}$ is bounded. Then by Bolzano-Weierstrass Theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $\alpha = \lim_{k\to\infty} x_{n_k}$. Since $a \leq x_{n_k} \leq b$, then $a \leq \alpha \leq b$. Since f is continuous at α, then we have $\lim_{k\to\infty} f(x_{n_k}) = f(\alpha)$. Also we have $|f(x_{n_k})| > n_k$, then $\lim_{k\to\infty} |f(x_{n_k})| = \infty$. Contradiction. Thus f is bounded on $[a, b]$.

Now, Let $m = \inf\{f(x) : x \in [a, b]\}$, then m is finite. For each $n \in \mathbb{N}$, $m + \frac{1}{n}$ is not a lower bound for $\{f(x) : x \in [a, b]\}$. Then there exists $x_n \in [a, b]$ such that $m \leq f(x_n) < m + \frac{1}{n}$. Hence $\lim_{n\to\infty} f(x_n) = m$. Now,$\{x_n\}$ is bounded in $[a, b]$. Then by Bolzano-Weierstrass Theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let $x_0 = \lim_{k\to\infty} x_{n_k} \in [a, b]$.

Since f is continuous at x_0, then we have $m = \lim_{k\to\infty} f(x_{n_k}) = f(x_0)$.

Hence $f(x_0) = m = \inf\{f(x) : x \in [a, b]\}$. Similarly, one can show the maximum value.

Theorem 0.3: *[The Intermediate Value Theorem]*

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$. Let α be a number between $f(a)$ and $f(b)$. [i.e. $f(a)<\alpha<f(b)$ or $f(b)<\alpha<f(a)$] Then there is a number $c\in(a,b)$ such that $f(c)=\alpha$.

Proof: Assume that $f(a)<\alpha<f(b)$. Let $E=\{x\in[a,b]: f(x)<\alpha\}$. Since $f(a)<\alpha$, then $a\in E$. Hence E is nonempty subset of $[a,b]$. Thus E is bounded. Then $c=\sup E$ exists and $c\in[a,b]$.

For each $n\in\mathbb{N}$, since $c-\dfrac{1}{n}$ is not an upper bound of E, then there exists $x_n\in E$ such that $c-\dfrac{1}{n}<x_n\le c$. Hence $\lim\limits_{n\to\infty}x_n=c$ and, since f is continuous at c, then $f(c)=\lim\limits_{n\to\infty}f(x_n)$. Now, $x_n\in E\ \forall,\ n\in\mathbb{N}$. Then $f(x_n)<\alpha$. Hence $f(c)=\lim\limits_{n\to\infty}f(x_n)\le\alpha$. Thus $f(c)\le\alpha \qquad (1)$.

Let $y_n=\min\{b,c+\dfrac{1}{n}\}$. Then $y_n\in[a,b]$, and $y_n\notin E\ \forall\ n\in\mathbb{N}$. Then $f(y_n)\ge\alpha$. Now, since $c\le y_n\le c+\dfrac{1}{n}$, then $\lim\limits_{n\to\infty}y_n=c$. Since f is continuous at c, then $f(c)=\lim\limits_{n\to\infty}f(y_n)\ge\alpha$. Thus $f(c)\ge\alpha \qquad (2)$. Then by (1) and (2) we have $f(c)=\alpha$. Since $f(a)<f(c)<f(b)$, then $c\neq a$ and $c\neq b$. Hence $c\in(a,b)$, and $f(c)=\alpha$.

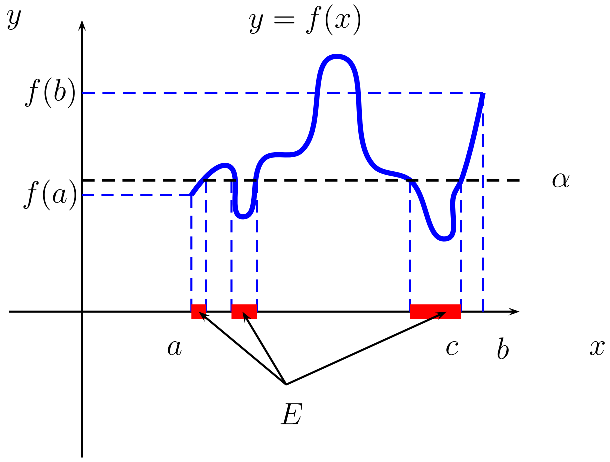

Figure 2:

Definition 0.4: Let $f:E\to\mathbb{R}$, be a function. We say f is ***uniformly continuous on*** E , if, for all $\epsilon>0$ there exists $\delta=\delta(\epsilon)>0$ such that if $x,y\in E$ and $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\epsilon$.

Example 0.3: Prove that $f(x)=x^2$ is uniformly continuous on $[a,b], a,b\in\mathbb{R}$.

Discussion: Given $\epsilon>0$, we want to find $\delta>0$ such that if $x,y\in[a,b]$, and $|x-y|<\delta\Rightarrow|x^2-y^2|<\varepsilon$. Now,

$$\begin{aligned}|x^2-y^2|&=|(x-y)(x+y)|\\&=|x-y||x+y|\\&\le|x-y|(|x|+|y|).\end{aligned}$$

Let $M = \max\{|a|, |b|\}$. Now, since $x, y \in [a, b]$, then $|x|, |y| \leq M$.

$$\begin{aligned}|x^2 - y^2| &= |(x-y)(x+y)| \\ &= |x-y||x+y| \\ &\leq |x-y|(|x|+|y|) \\ &\leq |x-y|(M+M) \\ &= 2M|x-y|\end{aligned}$$

If we choose $\delta = \dfrac{\epsilon}{2M}$.

Proof: Let $\epsilon > 0$ be given. Let $\delta = \dfrac{\epsilon}{2M}$.

$$\begin{aligned}\text{Now, if } x, y \in [a, b], \text{ with } |x-y| < \delta \Rightarrow |f(x) - f(y)| = |x^2 - y^2| &\leq 2M|x-y| \\ &< 2M\delta \\ &< 2M.\frac{\epsilon}{2M} \\ &= \epsilon.\end{aligned}$$

Hence f is uniformly continuous on $[a, b], a, b \in \mathbb{R}$.

Note 0.3: Let $f : E \to \mathbb{R}$, be a function. f is not uniformly continuous on E, if, there is $\epsilon_0 > 0$ such that for every $\delta > 0$ there are $x_\delta, y_\delta \in E$ such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.

Lemma 0.1: Let $f : E \to \mathbb{R}$, be a function. Then f is not uniformly continuous on E if and only if there exist $\epsilon_0 > 0$ and two sequence $\{x_n\}$ and $\{y_n\}$ in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Proof: ($\Rightarrow$) Suppose that f is not uniformly continuous ont E. Then there is $\epsilon_0 > 0$ such that for every $\delta > 0$ there are $x_\delta, y_\delta \in E$such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$. for each $n \in \mathbb{N}$, let $\delta = \dfrac{1}{n} > 0$, there are $x_n, y_n \in E$ such that $|x_n - y_n| < \dfrac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$. Hence we two sequence $\{x_n\}$ and $\{y_n\}$ in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

($\Leftarrow$) Suppose that there exist $\epsilon_0 > 0$ and two sequence $\{x_n\}$ and $\{y_n\}$ in E such that $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Let $\delta > 0$ be given. Since $\lim_{n\to\infty}(x_n - y_n) = 0$ there exist $N \in \mathbb{N}$ such that $n > N \Rightarrow |x_n - y_n| < \delta$. Thus $|x_{N+1} - y_{N+1}| < \delta$ and $|f(x_{N+1}) - f(y_{N+1})| \geq \epsilon_0$. Hence f is not uniformly continuous on E.

Example 0.4: Prove that $f(x) = x^2$ is not uniformly continuous on $[1, \infty)$.

Solution: Let $x_n = n$ and $y_n = n + \dfrac{1}{n}$. Now, $\{n\}, \{n + \dfrac{1}{n}\} \subset [1, \infty)$. Also, $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}\dfrac{1}{n} = 0$, $|f(x_n) - f(y_n)| = \left(n + \dfrac{1}{n}\right)^2 - n^2 = 2 + \dfrac{1}{n^2} > 2$. Hence f is not uniformly continuous on $[1, \infty)$.

Theorem 0.4: []

Let $a, b \in \mathbb{R}$ such that $a < b$. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$.

Proof: Suppose that f is not uniformly continuous on $[a,b]$. then there exist $\epsilon_0 > 0$ and two sequence $\{x_n\}$ and $\{y_n\}$ in $[a,b]$ such that $\lim\limits_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Since $\{x_n\} \subset [a,b]$, then it is bounded and hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ by Bolzano-Weierstrass Theorem. Since $[a,b]$ is closed then $x = \lim\limits_{k\to\infty} x_{n_k} \in [a,b]$. Also since $\lim\limits_{n\to\infty}(x_n - y_n) = 0$, then $x = \lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{k\to\infty} y_{n_k}$. Since f is continuous on $[a,b]$, then $f(x) = \lim\limits_{k\to\infty} f(x_{n_k}) = \lim\limits_{k\to\infty} f(y_{n_k})$. But $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$. Contradiction Hence f is uniformly continuous on $[a,b]$.

Definition 0.5: Let $E \subseteq \mathbb{R}$ and let $f : E \to \mathbb{R}$. We say that f is ***Lipschitz function*** on E if there is $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in E$.

Example 0.5: Prove that $f(x) = \sqrt{x}$ is Lipschitz function on $[1, \infty)$.

Solution: Since if $x, y \in [1, \infty)$, then $x, y \geq 1$ and hence $\sqrt{x} \geq 1$ and $\sqrt{y} \geq 1$. Thus $\sqrt{x} + \sqrt{y} \geq 2$, therefore $\dfrac{1}{\sqrt{x} + \sqrt{y}} \leq \dfrac{1}{2}$. Now, $|f(x) - f(y)| = \left|\sqrt{x} - \sqrt{y}\right| = \left|\dfrac{x - y}{\sqrt{x} + \sqrt{y}}\right| = \dfrac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \dfrac{1}{2}|x - y|$. Hence $f(x) = \sqrt{x}$ is Lipschitz function on $[1, \infty)$.

Theorem 0.5: *[]*

If $f : E \to \mathbb{R}$ is Lipschitz function on E, then f is uniformly continuous on E.

Proof: Since f is Lipschitz function on E, then there is $M > 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in E$. Let $\epsilon > 0$ be given. Choose $\delta = \dfrac{\epsilon}{M} > 0$, if $x, y \in E$ with $|x - y| < \delta$, then $|f(x) - f(y)| \leq M|x - y| < M.\dfrac{\epsilon}{M} = \epsilon$. Hence f is Lipschitz function on E.
