A Glimpse Into Topology of $\mathbb{R}$

Dr.Hamed Alsulami

November 30, 2016

Definition 0.1: An ***interval*** of $\mathbb{R}$ is a subset of $\mathbb{R}$ of one of the following types:

$$[a,b],(a,b),[a,b),(a,b],[c,\infty),(c,\infty),(-\infty,c],(-\infty,c),(-\infty,\infty)=\mathbb{R},$$

where $a,b,c\in\mathbb{R}$ and $a\le b$.

Note 0.1: A subset E of $\mathbb{R}$ is an interval if and only if $\forall\ x,y\in E,$ with $x\le y\Rightarrow[x,y]\subset E$.

Definition 0.2: A set E of real numbers is said to be ***open set*** if, for each $x\in E$ there is a number $\delta>0$ such that $(x-\delta,x+\delta)\subset E$.

Lemma 0.1: Let $a,b\in\mathbb{R}$, with $a\le b$. The intervals $(a,b),(a,\infty)$, and $(-\infty,a)$ are open sets.

Proof: **To show (a,b) is open:** Let $x\in(a,b)$. Let $\delta=\min\{x-a,b-x\}$, then $\delta\le x-a$ and $\delta\le b-x$. Now, $a=x-(x-a)\le x-\delta<x+\delta\le x+\delta=x+(b-x)=b$. Hence $(x-\delta,x+\delta)\subset(a,b)$. Thus (a,b) is open.

To show (a,∞) is open: Let $x\in(a,\infty)$. Let $\delta=x-a$, then $a=x-(x-a)=x-\delta<x+\delta=x+(x-a)=2x-a<\infty$. Hence $(x-\delta,x+\delta)\subset(a,\infty)$. Thus (a,∞) is open.

To show $(-\infty,a)$ is open: Let $x\in(-\infty,a)$. Let $\delta=a-x$, then $-\infty<2x-a=x-(a-x)=x-\delta<x+\delta=x+(a-x)=a$. Hence $(x-\delta,x+\delta)\subset(-\infty,a)$. Thus $(-\infty,a)$ is open.

Theorem 0.1: ***[Open set properties]***

(i) $\mathbb{R}$, and $\emptyset$ are open sets.

(ii) If $O_1,O_2,\cdots,O_n$ are open subsets of $\mathbb{R}$, then $\bigcap\limits_{k=1}^{n}O_k$ is an open set.

(iii) If $\{O_\alpha:\alpha\in I\}$ be a collection of open subsets of $\mathbb{R}$, then $\bigcup\limits_{\alpha\in I}O_\alpha$ is an open set.

Proof:

(i) For all $x\in\mathbb{R}$ and for all $\delta>0$, $(x-\delta,x+\delta)\subset\mathbb{R}$. Also since $\emptyset$ has no elements, then it is Open.

(ii) Let $x\in\bigcap\limits_{k=1}^{n}O_k$. Then $x\in O_k$ for every $k=1,2,\cdots,n$. Now, since every O_k is open for every $k=1,2,\cdots,n$, then there is $\delta_k>0$ such that $(x-\delta_k,x+\delta_k)\subseteq O_k$. Let $\delta=\min\{\delta_1,\delta_2,\cdots,\delta_n\}>0$. Since $\delta\le\delta_k$ for every $k=1,2,\cdots,n$, then $(x-\delta,x+\delta)\subseteq(x-\delta_k,x+\delta_k)\subseteq O_k$. Hence $(x-\delta,x+\delta)\subseteq\bigcap\limits_{k=1}^{n}O_k$. Thus $\bigcap\limits_{k=1}^{n}O_k$ is open.

(ii) Let $x \in \underset{\alpha\in I}{\cup} O_\alpha$. Then there exists $\alpha_0 \in I$ such that $x \in O_{\alpha_0}$. Now,O_{α_0} is open, then there exist $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq O_{\alpha_0}$. Now, $(x-\delta, x+\delta) \subseteq O_{\alpha_0} \subseteq \underset{\alpha\in I}{\cup} O_\alpha$. Hence $\underset{\alpha\in I}{\cup} O_\alpha$ is open.

Note 0.2: The intersection of an infinite number of open sets might not be open. For example, if $E_n = \left(1-\dfrac{1}{n}, 1+\dfrac{1}{n}\right)$ is open for every $n \in \mathbb{N}$. Now, $\overset{\infty}{\underset{n=1}{\cap}} E_n = \overset{\infty}{\underset{n=1}{\cap}} \left(1-\dfrac{1}{n}, 1+\dfrac{1}{n}\right) = \{1\}$ which is closed as we can show later.

Note 0.3: Let E be subset of $\mathbb{R}$. The ***complement*** of E, denoted by E^c $(\mathbb{R}-E)$, is the set of elements not in E.

Definition 0.3: A set E of real numbers is said to be ***closed set*** if E^c is open.

Lemma 0.2: Let $a, b \in \mathbb{R}$, with $a \leq b$. The intervals $[a,b], [a,\infty)$, $\{a\}$, and $(-\infty, a]$ are closed sets.

Proof: **To show** $[a,b]$ **is closed:** Since $[a,b]^c = (-\infty, a) \cup (b, \infty)$ which is open, then $[a,b]$ is closed set.

To show $[a,\infty)$ **is closed:** Since $[a,\infty)^c = (-\infty, a)$ which is open, then $[a,\infty)$ is closed set.

To show $\{a\}$ **is closed:** Since $\{a\}^c = (-\infty, a) \cup (a, \infty)$ which is open, then $\{a\}$ is closed set.

To show $(-\infty, a]$ **is closed:** Since $(-\infty, a]^c = (a, \infty)$ which is open, then $(-\infty, a]$ is closed set.

Theorem 0.2: *[Closed set properties]*

(i) $\mathbb{R}$, and $\emptyset$ are closed sets.

(ii) If $E_1, E_2, \cdots, E_n$ are closed subsets of $\mathbb{R}$, then $\overset{n}{\underset{k=1}{\cup}} E_k$ is a closed set.

(iii) If $\{E_\alpha : \alpha \in I\}$ be a collection of closed subsets of $\mathbb{R}$, then $\underset{\alpha\in I}{\cap} E_\alpha$ is a closed set.

Proof:

(i) Since $\mathbb{R}^c = \emptyset$ which is open, then $\mathbb{R}$ is closed. Also since $\emptyset^c = \mathbb{R}$ which is open, hence $\emptyset$ is closed.

(ii) Since, E_k is closed, for every $k = 1, 2, \cdots, n$, then E_k^c is open. Now, $(\overset{n}{\underset{k=1}{\cup}} E_k)^c = \overset{n}{\underset{k=1}{\cap}} E_k^c$ which is open. Hence $\cup_{k=1}^n E_k$ is a closed set.

(iii) Since, E_α is closed, for every $\alpha \in I$, then E_α^c is open. Now, $(\underset{\alpha\in I}{\cap} E_\alpha)^c = \underset{\alpha\in I}{\cup} E_\alpha^c$ which is open. Hence $\cap_{\alpha\in I} E_\alpha$ is a closed set.

Definition 0.4: Let E be subset of the real numbers. Let $x \in \mathbb{R}$.

(i) We say x is an ***interior point*** of E if there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subseteq E$. We denote the set of all interior points of E by E°, and it called the interior set of E.

(ii) We say x is a ***limit point*** of E if for every $\delta > 0$, $(x-\delta, x+\delta) \cap (E - \{x\}) \neq \emptyset$. We denote the set of all limit points of E by E', and it called the derived set of E.

Example 0.1: Let $E = (1, 5]$. Then every $x \in (1,5)$ is an interior point and every $x \in [1,5]$ is a limit point. Thus $E^\circ = (1,5)$, and $E' = [1,5]$.

Example 0.2: Let $E = \mathbb{Q}$. Let $x \in \mathbb{R}$. For every $\delta > 0$, the interval $(x-\delta, x+\delta)$ has irrational number in it, then $(x-\delta, x+\delta) \nsubseteq \mathbb{Q}$. Hence there are no interior points. Thus $\mathbb{Q}^\circ = \emptyset$. For every $\delta > 0$, the interval $(x-\delta, x+\delta)$ has rational number in it other than x, then $(x-\delta, x+\delta) \cap (\mathbb{Q} - \{x\}) \neq \emptyset$. Hence every point is a limit point. Thus $\mathbb{Q}' = \mathbb{R}$.

Theorem 0.3: *[Characterization of Closed sets]*
Let $E \subseteq \mathbb{R}$. Then the following are equivalent:

(i) E is a closed subset of $\mathbb{R}$

(ii) If $\{x_n\} \subset E$ is a sequence such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then $x \in E$.

Proof: $(i) \Rightarrow (ii)$ Suppose that E is a closed subset of $\mathbb{R}$ and $\{x_n\} \subset E$ is a sequence such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. We want to show that $x \in E$. Assume that $x \in E^c$, and since E is closed, then E^c is open. Hence there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset E^c$. Since $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that $n > N \Rightarrow |x_n - x| < \delta$. Thus $n > N \Rightarrow x - \delta < x_n < x + \delta$. Hence if $n > N \Rightarrow x_n \in (x-\delta, x+\delta) \subset E^c$. Contradiction since $x_n \in E$, for all $n \in \mathbb{N}$. Thus E is closed.
$(ii) \Rightarrow (i)$ Suppose that If $\{x_n\} \subset E$ is a sequence such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then $x \in E$. We want to show that E is closed. Let $x \in E^c$ if for all $\delta > 0$, $(x-\delta, x+\delta) \nsubseteq E^c$. Then $(x-\delta, x+\delta) \cap E \neq \emptyset$. Hence for each $n \geq 1$, there is $x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap E$. Thus $x_n \in E$ and $|x_n - x| < \frac{1}{n}$. Thus $\{x_n\} \subset E$ and $\lim_{n\to\infty} x_n = x \in E$. Then $x \in E$. Contradiction Hence there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset E^c$. Hence E^c is open. Thus E is closed.

Lemma 0.3: Let $E \subseteq \mathbb{R}$. Then E is closed if and only if E contains all of its limit points ($E' \subseteq E$).
Proof: $\Rightarrow$ Suppose E is closed and x is a limit point of E. We want to show that $x \in E$. Assume that $x \in E^c$. Since E is closed, then E^c is open and $x \in E^c$, then there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset E^c$. Hence $(x-\delta, x+\delta) \cap E = \emptyset$. Contradiction to the fact that x is a limit point of E. Hence $x \in E$.

$\Leftarrow$ Suppose E contains all of its limit points. We want to show that E is closed. Let $y \in E^c$. Then y is not a limit point of E. Hence there is $\delta > 0$ such that $(y-\delta, y+\delta) \cap E = \emptyset$. Hence $(y-\delta, y+\delta) \subset E^c$. Thus E^c is open. Hence E is closed.

Lemma 0.4: Let $E \subseteq \mathbb{R}$. Then E' is closed.
Proof: Let $x \in (E')^c$. Then x is not a limit point of E. Thus there exists $\delta > 0$ such that $(x-\delta, x+\delta) \cap (E - \{x\}) = \emptyset$. Let $y \in (x-\delta, x+\delta)$, and $y \neq x$. Since $(x-\delta, x+\delta)$ is open then there is $\beta > 0$ such that $(y-\beta, y+\beta) \subseteq (x-\delta, x+\delta)$. Now $(y-\beta, y+\beta) \cap (E - \{y\}) \subseteq (x-\delta, x+\delta) \cap (E - \{x\}) = \emptyset$, thus $(y-\beta, y+\beta) \cap (E - \{y\}) = \emptyset$. Therefore $y \notin E'$. Thus $(x-\delta, x+\delta) \subseteq (E')^c$. Thus $(E')^c$ is open. Hence E' is closed.

Definition 0.5: Let E be a subset of the real numbers. The collection of open sets $\{U_i\}_{i\in I}$ is said to be an ***open cover*** if $E \subset \cup_{i\in I} U_i$. If the number of sets in the collection I is finite, then the collection is said to be ***finite cover***

Example 0.3: Let $E = (1, 3)$. Let $U_n = (1 - \frac{1}{n}, 3 + \frac{1}{n})$ For every $n > 0$, the collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of E.

Definition 0.6: A subset E of the real numbers is said to be ***compact*** if every open cover has a finite subcover.

Example 0.4:

(i) Let $E = \{x_1, x_2, \cdots, x_n\}$ be a finite subset of $\mathbb{R}$. If $\{U_i\}_{i\in I}$ is an open cover of E, then for each x_k, for $k = 1, 2, \cdots, n$, there exists $U_{i_k} \subset \cup_{i\in I} U_i$ such that $x_k \in U_{i_k}$. Hence $E \subset \cup_{k=1}^{n} U_{i_k}$. Thus E is compact.

(ii) Let $B = [0, \infty)$. Then B is not compact. Let $U_n = (-1, n)$. The collection $\{U_n : n \in \mathbb{N}\}$ is an open cover of B. Now, any finite subcover will have the form $(-1, m)$ for some $m \in \mathbb{N}$ which will not cover B.

Theorem 0.4: ***[Heine-Borel Theorem]***

Let $a, b \in \mathbb{R}$ with $a \leq b$. Then $[a, b]$ is compact.

Proof: Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of $[a, b]$. Let $E = \{x \in [a, b] \mid [a, x]$ is covered by finite sets in $\mathcal{U}\}$. Now $a \in E$ and $E \subseteq [a, b]$, then $\sup E$ exists and in $\mathbb{R}$. Let $m = \sup E$, then $a \leq m \leq b$. Since $m \in [a, b] \subset \cup_{i\in I} U_i$, then there exists $U_{i_0} \in \mathcal{U}$ such that $m \in U_{i_0}$. Now, U_{i_0} is open , then there is a $\delta > 0$ such that $(m - \delta, m + \delta) \subseteq U_{i_0}$. Since $m - \delta$ is not an upper bound of E, then there exists $x \in E$ such that $m - \delta < x$. Now, $[a, x]$ is covered by finite sets in $\mathcal{U}$. Thus there exist $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ such that $[a, x] \subseteq \cup_{k=1}^{n} U_{i_k}$. Also $[x, m + \dfrac{\delta}{2}] \subseteq (m - \delta, m + \delta) \subseteq U_{i_0}$. Hence $[a, m + \dfrac{\delta}{2}] \subseteq U_{i_0} \cup \cup_{k=1}^{n} U_{i_k}$. Thus $[a, m + \dfrac{\delta}{2}]$ is covered by finite sets in $\mathcal{U}$, hence if $m + \dfrac{\delta}{2} \leq b$, then $m + \dfrac{\delta}{2} \in E$. Contradiction since $\sup E = m < m + \dfrac{\delta}{2} \in E$. Hence $b < m + \dfrac{\delta}{2}$ and for all $0 < \epsilon < \dfrac{\delta}{2}$, we have $b < m + \epsilon$. Hence $b \leq m$ and $m \leq b$. Thus $m = b$. Therefor $[a, b] \subseteq U_{i_0} \cup \cup_{k=1}^{n} U_{i_k}$. Hence $[a, b]$ is compact.

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Lemma 0.5: Let E be subset of $\mathbb{R}$. Then E is compact if and only if E is closed and bounded.

Proof: $\Rightarrow$ Suppose that E is compact. Then every open cover has a finite subcover. Let $U_n = (-n, n)$ which is open set in $\mathbb{R}$. Now, $E \subset \mathbb{R} = \cup_{n=1}^{\infty} U_n$. Hence $\{U_n : n \geq 1\}$ is an open cover and since E is compact then there exist $m \in \mathbb{N}$ such that $E \subset \cup_{n=1}^{m} U_n = (-m, m)$. Hence E is bounded. Let $y \in E^c$. Let $U_n = (-\infty, y - \dfrac{1}{n}) \cup (y + \dfrac{1}{n}, \infty)$. Then U_n is open set and $E \subset \mathbb{R} - \{y\} = \cup_{n=1}^{\infty}$. Since E is compact then there exist $m \in \mathbb{N}$ such that $E \subset \cup_{n=1}^{m} U_n = U_m$. Now $(y - \dfrac{1}{m}, y + \dfrac{1}{m}) \subset [y - \dfrac{1}{m}, y + \dfrac{1}{m}] = U_m^c \subset E^c$. Hence E^c is open and hence E is closed.

$\Leftarrow$ Suppose that E is closed and bounded. Since E is bounded then $m = \inf E \in \mathbb{R}$ and $M = \sup E \in \mathbb{R}$. and $E \subseteq [m, M]$. Now $[m, M] = E \cup ([m, M] - E)$ and since E is closed, then $E^c = \mathbb{R} - E$ is open set. Also notice that $[m, M] - E \subseteq E^c$. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of E. Hence $E \subset \cup_{i\in I}$. Then $[m, M] = E \cup ([m, M] - E) \subset \cup_{i\in I} U_i \cup E^c$. Since $[m, M]$ is compact, then there exist $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}, E^c$ such that $[m, M] = E \cup ([m, M] - E) \subset \cup_{k=1}^{n} U_{i_k} \cup E^c$. Since $E \cap ([m, M] - E) = \emptyset$ and $[m, M] - E \subseteq E^c$, then $E \subseteq \cup_{k=1}^{n} U_{i_k}$. Hence E is compact.

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