The Fundamental Theorem of Calculus

Dr.Hamed Al-Sulami

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Theorem 4.1: *[]*

Let $f : [a, b] \to \mathbb{R}$ be bounded function on $[a, b]$ and let $a < c < b$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof: ($\Rightarrow$) This left for you as homework.

($\Leftarrow$) Suppose f is integrable on $[a, c]$ and $[c, b]$. Let $\epsilon > 0$ be given. Since f is integrable on $[a, c]$ and $[c, b]$, there exist partitions P_1 of $[a, c]$ and P_2 of $[c, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \qquad \text{and} \qquad U(f, P_2) - L(f, P_2) < \frac{\epsilon}{2}.$$

Let $P_\epsilon = P_1 \cup P_2$, then P_ϵ is a partition of $[a, b]$. Also $U(f, P_\epsilon) = U(f, P_1) + U(f, P_2)$ and $L(f, P_\epsilon) = L(f, P_1) + L(f, P_2)$. Now,

$$U(f, P_\epsilon) - L(f, P_\epsilon) = [U(f, P_1) + U(f, P_2)] - [L(f, P_1) + L(f, P_2)] = [U(f, P_1) - L(f, P_1)] + [U(f, P_2) - L(f, P_2)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence f is integrable on $[a, b]$.

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Theorem 4.2: ***[The Fundamental Theorem of Calculus (first form)]***

Let $f : [a, b] \to \mathbb{R}$ be integrable function on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $F'(x) = f(x)$ for all $x \in (a, b)$. Then $\int_a^b f(x)\ dx = F(b) - F(a)$.

Proof: Let $\epsilon > 0$ be given. Since $F' = f$ is integrable on $[a, b]$, then there exist a partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a, b]$ such that

$$U(F', P) - L(F', P) = U(f, P) - L(f, P) < \epsilon.$$

Now, since F is continuous on $[a, b]$, then F is continuous on each interval $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$. Hence by the Mean Value Theorem there exist $t_k \in [x_{k-1}, x_k]$ such that

$$F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1}) = f(t_k)\triangle x_k.$$

Also we have

$$m_k \le f(t_k) \le M_k,$$

then

$$m_k \triangle x_k \le f(t_k)\triangle x_k \le M_k \triangle x_k.$$

Hence

$$m_k \triangle x_k \le [F(x_k) - F(x_{k-1})] \le M_k \triangle x_k.$$

Therefore

$$\sum_{k=1}^{n} m_k \triangle x_k \le \sum_{k=1}^{n} [F(x_k) - F(x_{k-1})] \le \sum_{k=1}^{n} M_k \triangle x_k.$$

Hence

$$L(F', P) = L(f, P) \le [F(b) - F(a)] \le U(f, P) = U(F', P).$$

Thus

$$-U(f, P) \le -[F(b) - F(a)] \le -L(f, P) \qquad (1).$$

Also

$$L(f, P) \le \int_a^b f \le U(f, P) \qquad (2).$$

Hence by adding (1) and (2) we get

$$-U(f, P) + L(f, P) \le \int_a^b f - [F(b) - F(a)] \le U(f, P) - L(f, P).$$

Therefore

$$-\epsilon < -(U(f, P) - L(f, P)) \le \int_a^b f - [F(b) - F(a)] \le U(f, P) - L(f, P) < \epsilon \Rightarrow \left| \int_a^b f - [F(b) - F(a)] \right| < \epsilon.$$

Hence $\int_a^b f(x)\, dx = F(b) - F(a)$.

Theorem 4.3: ***[The Fundamental Theorem of Calculus (second form)]***

Let $f : [a, b] \to \mathbb{R}$ be integrable function on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt$ for $x \in [a, b]$, then F is continuous on $[a, b]$. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c and

$$F'(c) = f(c).$$

Proof: Let $\epsilon > 0$ be given. Since f is bounded on $[a, b]$, then there exist a partition $M > 0$ such that

$$|f(x)| < M \text{ for } x \in [a, b].$$

Now, let $\delta = \frac{\epsilon}{M}$.

$$
\begin{aligned}
\text{If } x, y \in [a,b] \text{ such that } |x-y| < \delta \Rightarrow |F(x) - F(y)| &= \left| \int_a^x f(t)\ dt - \int_a^y f(t)\ dt \right| \\
&= \left| \int_x^y f(t)\ dt \right| \qquad \text{assume } x < y \\
&\le \int_y^x |f(t)|\ dt \\
&< \int_y^x M\ dt \\
&= M(x-y) \\
&= M|x-y| \\
&< M\frac{\epsilon}{M} = \epsilon.
\end{aligned}
$$

Thus F is uniformly continuous on $[a,b]$. Now, if f is continuous at $c \in [a,b]$, then there exist $\delta > 0$ such that

$$\text{if } x \in [a,b] \text{ and } |x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon.$$

Now,

$$
\begin{aligned}
\text{If } x \in [a,b] \text{ such that } 0 < |x-c| < \delta \Rightarrow \left| \frac{F(x)-F(c)}{x-c} - f(c) \right| &= \left| \frac{\int_a^x f(t)\ dt - \int_a^c f(t)\ dt}{x-c} - f(c) \right| \\
&= \left| \frac{\int_c^x f(t)\ dt}{x-c} - f(c) \right| \qquad \text{assume } c < x \\
&= \left| \frac{\int_c^x f(t)\ dt - \int_c^x f(c)\ dt}{x-c} \right| \qquad \text{note: } f(c) = \frac{\int_c^x f(c)\ dt}{x-c} \\
&= \left| \frac{\int_c^x [f(t) - f(c)]\ dt}{x-c} \right| \\
&\le \frac{\int_c^x |f(t) - f(c)|\ dt}{|x-c|} \\
&< \frac{\int_c^x \epsilon\ dt}{|x-c|} \\
&= \frac{\epsilon(x-c)}{|x-c|} \qquad \text{but } x - c = |x-c| \\
&< \frac{\epsilon\cancel{|x-c|}}{\cancel{|x-c|}} = \epsilon.
\end{aligned}
$$

Hence F is differentiable at c with $F'(c) = f(c)$.