The Completeness Property of $\mathbb{R}$

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2.1 Supremum and Infimum of a Set

Definition 2.1: Let A be a nonempty subset of $\mathbb{R}$.

(a) The set A is said to be ***bounded above*** if there exists a number $u \in \mathbb{R}$ such that $a \leq u$ for all $a \in A$. Each such number u is called an ***upper bound*** of A.

(b) The set A is said to be ***bounded below*** if there exists a number $w \in \mathbb{R}$ such that $w \leq a$ for all $a \in A$. Each such number w is called an ***lower bound*** of A.

(c) A set is said to be ***bounded*** if it is both bounded above and bounded below. A set is said to be ***unbounded*** if it is not bounded.

Example 2.1: Determine whether the given set bounded above, bounded below, and bounded.

1. $A = \left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$
2. $B = \{x \in \mathbb{R} : x < 7\}$
3. $C = \{x \in \mathbb{R} : x > 2\}$

Solution:

1. Since $\dfrac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, then A is bounded above. Also, since $0 < \dfrac{1}{n}$ for all $n \in \mathbb{N}$, then A is bounded below. Hence A is bounded set.
2. Since $x < 7$ for all $x \in B$, then B is bounded above. B is not bounded below since for all $m \in \mathbb{Z}$ such that $m < 7$, then $m \in B$. Thus B is unbounded.
3. Since $2 < x$ for all $x \in C$, then C is bounded below. C is not bounded above since for all $m \in \mathbb{Z}$ such that $m > 2$ then $m \in C$. Thus C is unbounded.

Definition 2.2: Let A be a nonempty subset of $\mathbb{R}$.

(a) If A is bounded above, then a number $u \in \mathbb{R}$ is said to be ***supremum (least upper bound)*** of A if it satisfies the conditions:

(1) u is an upper bound of A (i.e. $a \leq u$ for all $a \in A$.), and

(2) If v is any upper bound of A then $u \leq v$.

We will denote the supremum of A by $\sup A$.

(b) If A is bounded below, then a number $w \in \mathbb{R}$ is said to be ***infimum (greatest lower bound)*** of A if it satisfies the conditions:

(1) w is a lower bound of A (i.e. $w \leq a$ for all $a \in A$.), and

(2) If t is any lower bound of A, then $t \leq w$.

We will denote the infimum of A by $\inf A$.

Example 2.2: Find sup and inf for each set if they exist.

1. $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
2. $B = \{x \in \mathbb{R} : x < 7\}$
3. $C = \{x \in \mathbb{R} : x > 2\}$

Solution:

1. We will prove later that $\sup A = 1$ and $\inf A = 0$.
2. $\sup B = 7$ and $\inf B = -\infty$.
3. $\sup C = \infty$ and $\inf B = 2$.

Lemma 2.1: Let A be nonempty subset of $\mathbb{R}$. Let $u \in \mathbb{R}$. Then $u = \sup A$ if and only if u is an upper bound for A and for each $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$.

Proof: ($\Rightarrow$) Suppose that $u = \sup A$. Then u is an upper bound of A. Let $\epsilon > 0$ be given since $u - \epsilon < u$ and u is the least upper bound, then $u - \epsilon$ is not an upper bound for A. Hence there exists $a_\epsilon \in A$ such that $a_\epsilon > u - \epsilon$.
($\Leftarrow$) Suppose that u is an upper bound for A and for each $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$. Let v be any upper bound for A. Assume that $v < u$. Let $\epsilon_0 = u - v > 0$. Now there exist $a_{\epsilon_0} \in A$ such that $a_{\epsilon_0} > u - \epsilon_0 = u - (u - v) = u - u + v = v$. Thus $v < a_{\epsilon_0}$. Contradiction. Hence $u \leq v$. Therefore $u = \sup A$.

Lemma 2.2: Let A be nonempty subset of $\mathbb{R}$. Let $u \in \mathbb{R}$. Then $w = \inf A$ if and only if w is a lower bound for A and for each $\epsilon > 0$ there exists $a_\epsilon \in A$ such that $a_\epsilon < w + \epsilon$.

Definition 2.3: Let F be an ordered field. We say that F is complete if for any nonempty subset A of F that is bounded above, then $\sup A \in F$.

Completeness Axiom 2.1: Every nonempty subset of $\mathbb{R}$ that has an upper bound also has a supremum in $\mathbb{R}$.

Definition 2.4: Let A, B be two nonempty subsets of $\mathbb{R}$, and $c \in \mathbb{R}$. We define the following sets:

1. $A + B = \{a + b : a \in A \text{ and } b \in B\}$
2. $A - B = \{a - b : a \in A \text{ and } b \in B\}$

3. $A + c = \{a + c : a \in A\}$
4. $cA = \{ca : a \in A\}$
5. $AB = \{ab : a \in A \text{ and } b \in B\}$

Note that $2A$ may not be the same as $A+A$. For, example if $A = \{-2, 2\}$, then $2A = \{-4, 4\}$ and $A+A = \{-4, 0, 4\}$, and hence $2A \neq A + A$.

Corollary 2.1: Every nonempty subset A of $\mathbb{R}$ that is bounded below has a infimum in $\mathbb{R}$.

Proof: Let $-A = \{-a : a \in A\}$. Since A is bounded below there is $m \in \mathbb{R}$ such that $m \leq a$ for all $a \in A$. Hence $-a \leq -m$ for all $a \in A$. Thus the set $-A$ is bounded above. Then by Completeness axiom $\sup(-A)$ exist and is a real number. Let $\alpha = \sup(-A)$ Now, $-a \leq \alpha$ for all $a \in A$. Hence $-\alpha \leq a$ for all $a \in A$. Thus $-\alpha$ is a lower bound for A. Let β be a lower bound for A. hence $\beta \leq a$ for all $a \in A$. Thus $-a \leq -\beta$ for all $a \in A$. Thus $-\beta$ is an upper bound for $-A$, but $\alpha = \sup(-A)$ and hence $\alpha \leq -\beta$ and therefore $\beta \leq -\alpha$. Thus $\inf A = -\alpha \in \mathbb{R}$. Hence $\inf A = -\sup(-A)$.

Lemma 2.3: Let A, B be two nonempty subsets of $\mathbb{R}$, and $c \in \mathbb{R}$. Then

a. $\sup(A + B) = \sup A + \sup B$,

b. $\inf(cA) = c \inf A$, if $c > 0$, and $\inf(cA) = c \sup A$, if $c < 0$.

c. $\sup(A + c) = \sup A + c$

d. $\inf(A - B) = \inf A - \sup B$.

Proof:

a. Let $\alpha = \sup A$, and $\beta = \sup B$. we want to show that $\sup(A + B) = \alpha + \beta$. Since $\alpha = \sup A$, then $a \leq \alpha$, for all $a \in A$. Also, since $\beta = \sup B$, then $b \leq \beta$, for all $b \in B$. By adding the inequalities we have $a + b \leq \alpha + \beta$, for all $a \in A$, and $b \in B$. Thus $\alpha + \beta$ is an upper bound for the set $A + B$. Let γ be an upper bound for $A + B$. Suppose that $\gamma < \alpha + \beta$. Then $\gamma - \beta < \alpha$ and since $\alpha = \sup A$, hence $\gamma - \beta$ is not an upper bound for A. Then there exist $a_0 \in A$ such that $\gamma - \beta < a_0$. Thus $\gamma - a_0 < \beta$ and since $\beta = \sup B$, then $\gamma - a_0$ is not an upper bound for B. Then there exist $b_0 \in B$ such that $\gamma - a_0 < b_0$. Therefore $\gamma < a_0 + b_0$. Contradiction since γ is an upper bound for $A + B$. Thus $\alpha + \beta \leq \gamma$. Hence $\sup(A + B) = \alpha + \beta = \sup A + \sup B$.

b. Let $\alpha = \inf A$, and $\beta = \sup A$, we want to show that $\inf(cA) = c\alpha$, if $c > 0$ and $\inf(cA) = c\beta$, if $c < 0$. Since $\alpha = \inf A$, then $\alpha \leq a$, for all $a \in A$.

  - Case I: if $c > 0$, then $c\alpha \leq ca$, for all $a \in A$. Hence $c\alpha$ is a lower bound for cA. Let γ be a lower bound for cA. Assume that $c\alpha < \gamma$, then $\alpha < \dfrac{\gamma}{c}$, and since $\alpha = \inf A$, then $\dfrac{\gamma}{c}$ is not a lower bound for A. Hence there is $a_0 \in A$ such that $a_0 \leq \dfrac{\gamma}{c}$. Thus $ca_0 \leq \gamma$. Contradiction since γ is a lower bound for cA. Thus $\gamma \leq c\alpha$. Hence $\inf(cA) = c\alpha = c \inf A$.
  - Case II: if $c < 0$, since $\beta = \sup A$, then $a \leq \beta$, for all $a \in A$. Hence $c\beta \leq ca$, for all $a \in A$. Thus $c\beta$ is a lower bound for cA. Let δ be a lower bound for cA. Assume that $c\beta < \delta$, then $\dfrac{\delta}{c} < \beta$. Since $\beta = \sup A$, then $\dfrac{\delta}{c}$ is not an upper bound for A. Then there exist $a_0 \in A$ such that $\dfrac{\delta}{c} < a_0$ and hence $ca_0 < \delta$. Contradiction since δ is a lower bound for cA. Thus $\delta \leq c\beta$. Therefore $\inf(cA) = c\beta = c \sup A$.

c. Let $\alpha = \sup A$, and $c \in \mathbb{R}$. we want to show that $\sup(A+c) = \alpha + c$. Since $\alpha = \sup A$, then $a \leq \alpha$, for all $a \in A$. Hence $a + c \leq \alpha + c$, for all $a \in A$. Thus $\alpha + c$ is an upper bound for $A + c$. Let $\epsilon > 0$ be given. Since $\alpha = \sup A$, then there exist $a_\epsilon \in A$ such that $\alpha - \epsilon < a_\epsilon$, hence $a_\epsilon + c \in A + c$ and $\alpha + c - \epsilon < a_\epsilon + c$. Therefore $\sup(A+c) = \alpha + c = \sup A + c$.

d. Since $A - B = A + (-B)$, then using argument similar to part (a) we have $\inf(A+B) = \inf A + \inf B$ and using part (b) $\inf(-A) = -\sup A$. Hence $\inf(A-B) = \inf(A+(-B)) = \inf A + \inf(-B) = \inf A - \sup B$.

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2.2 Archimedean Property

Theorem 2.1: *[Archimedean Property]*

If $x \in \mathbb{R}$, then there exist $n_x \in \mathbb{N}$ such that $x < n_x$.

Proof: Suppose that $n \leq x$, for all $n \in \mathbb{N}$. Hence $\mathbb{N}$ is bounded above by x. Then by the Completeness Axiom $\sup \mathbb{N}$ is a real number. Let $u = \sup \mathbb{N}$. Now, $u - 1$ is not an upper bound for $\mathbb{N}$, then there exist $m \in \mathbb{N}$ such that $u - 1 < m$ and hence $u < m + 1$. Contradiction, since $u = \sup \mathbb{N}$ and $m + 1 \in \mathbb{N}$. There for for each $x \in \mathbb{R}$ there exist $n_x \in \mathbb{N}$ such that $x < n_x$.

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Corollary 2.2:

1. If $a > 0$, there exist $n \in \mathbb{N}$ such that $\frac{1}{n} < a$.
2. If $a > 0$ and $b > 0$, there exist $n \in \mathbb{N}$ such that $na > b$.

Proof:

1. If $a > 0$, then $\frac{1}{a} > 0$ and hence by Archimedean Property, there exist $n \in \mathbb{N}$ such that $\frac{1}{a} < n$. Thus $\frac{1}{n} < a$.
2. If $a > 0$ and $b \in \mathbb{R}$, then $\frac{b}{a} \in \mathbb{R}$ and hence by Archimedean Property, there exist $n \in \mathbb{N}$ such that $\frac{b}{a} < n$. Thus $b < na$.

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Example 2.3: Prove the following

1. $\sup\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 1$ and $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$.
2. $\sup\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = 1$ and $\inf\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = \frac{1}{2}$.

Solution:

1. Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Clearly that for all $n \in \mathbb{N}$, we have $n \geq 1$ and hence $\frac{1}{n} \leq 1$. Thus 1 is an upper bound for A. Let $\epsilon > 0$ be given. Since $1 - \epsilon < 1 \in A$ then $\sup A = 1$. Clearly that $0 < \frac{1}{n}$, for all $n \in \mathbb{N}$. Hence 0 is a lower bound for A.Now, suppose $w = \inf A$, clearly $0 \leq w$. Let $\epsilon > 0$ be given. Then there exist $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Now, $0 \leq w \leq \frac{1}{n_0} < \epsilon$. Thus $w = 0$.

2. Let $B = \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$. Clearly that for all $n \in \mathbb{N}$, we have $n < n+1$ and hence $\frac{n}{n+1} < 1$. Thus 1 is an upper bound for B. Let $\epsilon > 0$ be given. There exist $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$ and hence $-\epsilon < -\frac{1}{n_0}$. Thus $1 - \epsilon < 1 - \frac{1}{n_0} = \frac{n_0 - 1}{n_0} \in B$. Thus $\sup B = 1$. Now, $1 \leq n$, and $n + 1 \leq 2n$. Hence $\frac{1}{2} \leq \frac{n}{n+1}$ for all $n \in \mathbb{N}$. Thus $\frac{1}{2}$ is a lower bound for B. For each $\epsilon > 0$, we have $\frac{1}{2} < \frac{1}{2} + \epsilon$. Then $\inf\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\} = \frac{1}{2}$.

Lemma 2.4: Let $x \in \mathbb{R}$. Then there exist $n \in \mathbb{Z}$ such that $n \leq x < n+1$.

Proof: By Archimedean Property, there exist $m \in \mathbb{N}$ such that $|x| < m$. Hence $-m < x < m$. The se $A_x = \{-m, -m+1, \ldots, 0, 1, \ldots, m-1, m\}$ is a finite set. The se $B_x = \{k : k \in A_x \text{ and } k \leq x\} \subset A_x$ is bounded above by x. Let $n = \sup B_x \in B_x$. Then $n \leq x$ and $n + 1 \notin B_x$. Hence $n \leq x < n+1$.

2.3 The Existence of the Square Root

Theorem 2.2: ***[The Existence of the Square Root]***

There exist a positive real number x such that $x^2 = 2$.

Proof: Let $A = \{a \in \mathbb{R} : a \geq 0 \text{ and } a^2 \leq 2\}$. The A is nonempty subset of $\mathbb{R}$ $(0, 1 \in A.)$ Let $u > 2$, then $u^2 > 4$. Hence $u \notin A$. Thus $a < 2$ for all $a \in A$. Thus A is bounded above. Then $\sup A \in \mathbb{R}$. Let $x = \sup A$. Then $x \geq 1$, because $1 \in A$. Let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ such that $\frac{4x}{n} < \epsilon$ and $\frac{1}{n} < x$. Then $0 < x - \frac{1}{n} < x < x + \frac{1}{n}$ and hence $(x - \frac{1}{n})^2 < x^2 < (x + \frac{1}{n})^2$. Now, $x - \frac{1}{n}$ is not an upper bound for A. Then there exist $a \in A$ such that $x - \frac{1}{n} < a$ and hence $(x - \frac{1}{n})^2 < a^2 \leq 2$. Also $x + \frac{1}{n} \notin A$, hence $(x + \frac{1}{n})^2 > 2$. Thus $(x - \frac{1}{n})^2 < 2 < (x + \frac{1}{n})^2$. Now, we have $(x - \frac{1}{n})^2 < x^2 < (x + \frac{1}{n})^2$, and $-(x + \frac{1}{n})^2 < -2 < -(x - \frac{1}{n})^2$. By adding the last two inequalities we have $(x - \frac{1}{n})^2 - (x + \frac{1}{n})^2 < x^2 - 2 < (x + \frac{1}{n})^2 - (x - \frac{1}{n})^2$. Thus $(2x)(\frac{-2}{n}) < x^2 - 2 < (2x)(\frac{2}{n})$. Therefore $-\frac{4x}{n} < x^2 - 2 < \frac{4x}{n}$. Hence $|x^2 - 2| < \frac{4x}{n} < \epsilon$. Since for each $\epsilon > 0$ we have $|x^2 - 2| < \epsilon$, then $x^2 - 2 = 0$. Thus $x^2 = 2$.

2.4 Density of Rational and Irrational Numbers in $\mathbb{R}$

Theorem 2.3: ***[Density of $\mathbb{Q}$]***

If $a, b \in \mathbb{R}$ with $a < b$, then there exist a rational number $r \in \mathbb{Q}$ such that $a < r < b$.

Proof: Since $a < b$, then $b - a > 0$ and $1 > 0$, using Archimedean Property, there exist $n \in \mathbb{N}$ such that $n(b-a) > 1$. Now, $nb > na + 1$ and since $na + 1 \in \mathbb{R}$, there exist $m \in \mathbb{Z}$ such that $m \leq na + 1 < m + 1$. $m \leq na + 1 < nb$ and $na + 1 < m + 1$, and hence $na < m$. Hence $na < m < nb$ and therefore $a < \frac{m}{n} < b$. Let $r = \frac{m}{n} \in \mathbb{Q}$, then $a < r < b$.

The above theorem saying that between any two real numbers there is a rational number. Using this theorem we can prove that any real number can be approximated by a rational number. Another version of the above theorem is the following:

Theorem 2.4: ***[Approximation of $\mathbb{R}$ by $\mathbb{Q}$]***

Let $a \in \mathbb{R}$. For each $\epsilon > 0$ there exist $r_\epsilon \in \mathbb{Q}$ such that $|a - r_\epsilon| < \epsilon$.

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \epsilon$. Now, $Na \in \mathbb{R}$, then there exist $m \in \mathbb{Z}$ such that $m \leq Na < m+1$. Hence $\dfrac{m}{N} \leq a < \dfrac{m}{N} + \dfrac{1}{N}$. Thus $0 \leq a - \dfrac{m}{N} < \dfrac{1}{N}$. Hence $|a - \dfrac{m}{N}| < \dfrac{1}{N} < \epsilon$. Let $r_\epsilon = \dfrac{m}{N} \in \mathbb{Q}$. Then for each $\epsilon > 0$ there exist $r_\epsilon \in \mathbb{Q}$ such that $|a - r_\epsilon| < \epsilon$.

Theorem 2.5: ***[Density of $\mathbb{Q}^c$]***

If $a, b \in \mathbb{R}$ with $a < b$, then there exist a irrational number $z \in \mathbb{Q}^c$ such that $a < z < b$.

Proof: Since $a < b$, then $\sqrt{2} > 0$ then $a\sqrt{2} < b\sqrt{2}$. Using the Density Theorem of $\mathbb{Q}$ there exist $r \in \mathbb{Q}$ such that $a\sqrt{2} < r < b\sqrt{2}$. Hence $a < r\sqrt{2} < b$. Let $z = r\sqrt{2} \in \mathbb{Q}^c$, then $a < z < b$.

