# The Algebraic and Order Properties of $\mathbb{R}$

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**Definition 1.1:** There are two operations on $\mathbb{R}$ **addition** and **multiplication**. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (*commutative law for addition*)

(A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$(*associative law for addition*)

(A3) there exists an element $0 \in \mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$ (*existence of a zero element*)

(A4) for all $a \in \mathbb{R}$ there exists an element $-a \in \mathbb{R}$ such that $a+(-a) = (-a)+a = 0$ (*existence of negative elements* )

(M1) $ab = ba$ for all $a, b \in \mathbb{R}$ (*commutative law for multiplication*)

(M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$(*associative law for multiplication*)

(M3) there exists an element $0 \neq 1 \in \mathbb{R}$ such that $1a = a1 = a$ for all $a \in \mathbb{R}$ (*existence of a unit element*)

(M4) for all $0 \neq a \in \mathbb{R}$ there exists an element $\frac{1}{a} \in \mathbb{R}$ such that $a\frac{1}{a} = \frac{1}{a}a = 1$ (*existence of reciprocals* )

(D) $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbb{R}$ (*distributive law of multiplication over addition* )

**Theorem 1.1:** ***[uniqueness of units and inverses ]***

(i) If $z, a \in \mathbb{R}$ with $z + a = a$, then $z = 0$.

(ii) If $0 \neq b, u \in \mathbb{R}$ with $bu = b$, then $u = 1$.

(iii) If $a \in \mathbb{R}$, then $a0 = 0$.

(iv) If $0 \neq a, b \in \mathbb{R}$ with $ab = 1$, then $b = \frac{1}{a}$.

(v) If $ab = 0$, then either $a = 0$ or $b = 0$.

***Proof:***

$$(i) z = z + 0 = z + (a + (-a)) = (z + a) + (-a) = a + (-a) = 0.$$

$$(ii) u = u1 = u(b\frac{1}{b}) = (ub)\frac{1}{b} = b(\frac{1}{b}) = 1.$$

$$(iii) a0 + a = a0 + a1 = a(0 + 1) = a1 = a, \text{ then by } (i) \text{ we have } a0 = 0.$$

$$(iv) b = 1b = ((\frac{1}{a})a)b = (\frac{1}{a})(ab) = (\frac{1}{a})1 = (\frac{1}{a}).$$

$$(v) \text{ suppose } b \neq 0, \text{ then } a = a1 = a(b(\frac{1}{b})) = (ab)(\frac{1}{b}) = 0(\frac{1}{b}) = 0.$$

## 1.1 Subsets of the Real Numbers

- **The set of natural numbers:**
  We denote it by $\mathbb{N}$ and $\mathbb{N} := \{1, 2, 3, ...\}$. A natural number $n$ is even if it has the form $n = 2l$ for some $l \in \mathbb{N}$. A natural number $n$ is odd if it has the form $n = 2k + 1$ for some $k \in \mathbb{N}$.

- **The set of integer numbers:**
  We denote it by $\mathbb{Z}$ and $\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}$.

- **The set of rational numbers:**
  We denote it by $\mathbb{Q}$ and $\mathbb{Q} := \left\{\frac{n}{m} | n, m \in \mathbb{Z} \text{ and } m \neq 0\right\}$.
  Note that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

- **The set of irrational numbers:** It may not look obvious, but there are some real numbers not rational numbers. For example, if $p$ is a prim number, then there is no rational number $r$ such that $r^2 = p$. We denote the set of irrational numbers by $\mathbb{Q}'$ and $\mathbb{Q}' := \{x \in \mathbb{R} | x \notin \mathbb{Q}\}$.

***Theorem 1.2:*** *[$\sqrt{p}$ is not a rational number for any prim $p$.]*
Let $p$ is a prim number. Then there does not exist a rational number $r$ such that $r^2 = p$.
***Proof:*** Suppose that there is $n, m \in \mathbb{Z}$ such that $p = \left(\frac{n}{m}\right)^2$ and $(n, m) = 1$. Hence $pm^2 = n^2$ and $p|n^2$. Since $p$ is a prim, then $p|n$. Thus $n = pl$ for some $l \in \mathbb{N}$. Therefore $pm^2 = n^2 = p^2l^2$ and hence $m^2 = pl^2$. Thus $p|m^2$ and hence $p|m$. Thus $p|n$ and $p|m$. Contradiction since $(n, m) = 1$. Therefore there does not exist a rational number $r$ such that $r^2 = p$.

## 1.2 The Order properties of $\mathbb{R}$

***Definition 1.2:*** There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers**, that satisfies the following:

- If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$.
- If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$.
- If $a \in \mathbb{R}$, then exactly one of the following holds: $a \in \mathbb{P}$, $a = 0$, $-a \in \mathbb{P}$

***Note 1.1:*** Note that $\mathbb{R} = \mathbb{P} \cup \{0\} \cup \{-a : a \in \mathbb{P}\}$.
If $a \in \mathbb{P}$ we say $a$ is positive and write $a > 0$.
If $a \in \mathbb{P} \cup \{0\}$ we say $a$ is nonnegative and write $a \geq 0$.
If $-a \in \mathbb{P}$ we say $a$ is negative and write $a < 0$.
If $-a \in \mathbb{P} \cup \{0\}$ we say $a$ is nonpositive and write $a \leq 0$.

***Definition 1.3:*** Let $a, b \in \mathbb{R}$.

1. If $a - b \in \mathbb{P}$, then we write $a > b$ or $b < a$.
2. If $a - b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

**Theorem 1.3:** ***[Rules of Inequalities]***

Let $a, b, c \in \mathbb{R}$.

(a) If $a > b$ and $b > c$, then $a > c$.

(b) If $a > b$, then $a + c > b + c$.

(c) If $a > b$ and $c > 0$, then $ac > bc$.

(d) If $a > b$ and $c < 0$, then $ac < bc$.

***Proof:***

(a) If $a > b$ and $b > c$, then $a - b, b - c \in \mathbb{P}$ and $a - c = (a - b) + (b - c) \in \mathbb{P}$. Thus $a > c$.

(b) If $a > b$, then $a - b \in \mathbb{P}$. Now, $(a + c) - (b + c) = a - b \in \mathbb{P}$, thus $a + c > b + c$.

(c) If $a > b$ and $c > 0$, then $a - b, c \in \mathbb{P}$. Hence $ac - bc = (a - b)c \in \mathbb{P}$. Thus $ac > bc$.

(d) If $a > b$ and $c < 0$, then $a - b \in \mathbb{P}$ and $-c \in \mathbb{P}$. Now, $bc - ac = (a - b)(-c) \in \mathbb{P}$. Thus $ac < bc$.

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**Theorem 1.4:** ***[Positivity]***

Let $a, b \in \mathbb{R}$.

(i) $a^2 \geq 0$ for all $a \in \mathbb{R}$.

(ii) $1 > 0$.

(iii) If $n \in \mathbb{N}$, then $n > 0$.

(iv) If $a > 0$, then $\frac{1}{a} > 0$.

(v) If $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

(vi) If $a > 0$, then $0 < \frac{a}{2} < a$.

***Proof:***

(i) If $a \geq 0$, then $a.a > a.0$. Hence $a^2 > 0$. If $a < 0$, then $-a > 0$. Hence $-a. - a > -a.0$, so $a^2 > 0$.

(ii) we know by part (i) that $1^2 > 0$. Now $1 = 1^2 > 0$.

(iii) Using mathematical induction: by (ii) $1 > 0$. Now, suppose that $k \in \mathbb{N}$ and $k > 0$. Hence $k, 1 \in \mathbb{P}$ and thus $k + 1 = (k) + (1) \in \mathbb{P}$. Therefore $k + 1 > 0$. Thus for all $n \in \mathbb{N}$, $n > 0$.

(iv) Suppose $\frac{1}{a} \leq 0$. Then $a.\frac{1}{a} \leq a.0$. Hence $1 \leq 0$. contradiction. Thus $\frac{1}{a} > 0$.

(v) Since $0 < a < b$, then $\frac{1}{a} > 0$, $\frac{1}{b} > 0$, and $b - a > 0$. Now, $\frac{1}{a} - \frac{1}{b} = \frac{b - a}{ab} = \frac{1}{a}(b - a)\frac{1}{b} > 0$. Hence $0 < \frac{1}{b} < \frac{1}{a}$.

(vi) Since $0 < 1 < 2$, then by (v) we have $0 < \frac{1}{2} < 1$. Since $a > 0$, then $a.0 < a.\frac{1}{2} < a.1$ and hence $0 < \frac{a}{2} < a$.

 

**Theorem 1.5:** *[]*

Let $a \in \mathbb{R}$. If $0 \leq a < \epsilon$ for all $\epsilon > 0$, then $a = 0$.

***Proof:*** Suppose that $a > 0$. Choose $\epsilon_0 = \dfrac{a}{2} > 0$. Since $0 < \dfrac{a}{2} < a$, then $0 < \epsilon_0 < a$. Contradiction. Thus $a = 0$.

**Theorem 1.6:** *[]*

Let $a, b \in \mathbb{R}$. $ab > 0 \Leftrightarrow a > 0$, and $b > 0$ or $a < 0$, and $b < 0$.

***Proof:*** $(\Rightarrow)$ Suppose that $ab > 0$. Then $a \neq 0$ and $b \neq 0$. If $a > 0$, then $\dfrac{1}{a} > 0$ and hence $b = \dfrac{1}{a}ab > 0$. If $a < 0$, then $\dfrac{1}{a} < 0$ and hence $b = \dfrac{1}{a}ab < 0$.

$(\Leftarrow)$ Suppose $a > 0$, and $b > 0$ or $a < 0$, and $b < 0$. If $a > 0$, and $b > 0$, then $ab > 0$. If $a < 0$, and $b < 0$, then $ab > 0$.

**Theorem 1.7:** *[]*

Let $a, b \in \mathbb{R}$.

$$(i) \quad 0 < a \leq b \Leftrightarrow \sqrt{a} \leq \sqrt{b}$$

$$(ii) \quad \text{If } a > 0, b > 0, \text{ then } \sqrt{ab} \leq \frac{a+b}{2}$$

***Proof:***

$(i)(\Rightarrow)$ Suppose $a \leq b \Rightarrow 0 \leq b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a})$ and since $\sqrt{b} + \sqrt{a} \geq 0 \Rightarrow \sqrt{b} - \sqrt{a} \geq 0 \Rightarrow \sqrt{a} \leq \sqrt{b}$.

$(\Leftarrow)$ Suppose $\sqrt{a} \leq \sqrt{b} \Rightarrow \sqrt{b} - \sqrt{a} \geq 0$ and $\sqrt{b} + \sqrt{a} \geq 0 \Rightarrow b - a = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) \geq 0$

$(ii) 0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b \Rightarrow 2\sqrt{ab} \leq a + b \Rightarrow \sqrt{ab} \leq \dfrac{a+b}{2}$.

## 1.3 Absolute Value

**Definition 1.4:** The *absolute value* of a real number $a$, denoted by$|a|$, is defined by $|a| = \begin{cases} a, & \text{if } a \geq 0; \\ -a, & \text{if } a < 0. \end{cases}$

**Theorem 1.8:** ***[Absolute Value Properties]***

1. $|ab| = |a||b|$ for all $a \in \mathbb{R}$.
2. $|a|^2 = a^2$ for all $a \in \mathbb{R}$.
3. If $c \geq 0$, then $|a| \leq c \Leftrightarrow -c \leq a \leq c$.
4. $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.

***Proof:***

1. We prove this with cases

   - Case I: If $a = 0$ or $b = 0$, then $ab = 0$ and hence $|ab| = 0 = |a||b|$.

 

- Case II: If $a > 0$ and $b > 0$, then $ab > 0$. Thus $|ab| = ab$, $|a| = a$, $|b| = b$.
  Therefore $|ab| = ab = |a||b|$.
- Case III: If $a > 0$ and $b < 0$, then $ab < 0$. Thus $|ab| = -ab$, $|a| = a$, $|b| = -b$.
  Therefore $|ab| = -ab = a(-b) = |a||b|$.
- Case IV: If $a < 0$ and $b > 0$, then $ab < 0$. Thus $|ab| = -ab$, $|a| = -a$, $|b| = b$.
  Therefore $|ab| = -ab = (-a)b = |a||b|$.
- Case V: If $a < 0$ and $b < 0$, then $ab > 0$. Thus $|ab| = ab$, $|a| = -a$, $|b| = -b$.
  Therefore $|ab| = ab = (-a)(-b) = |a||b|$.

2. Since $a^2 \geq 0$, then $a^2 = |a^2| = |aa| = |a||a| = |a|^2$.

3. $(\Rightarrow)$ Suppose $|a| \leq c$. If $a = 0$, then $-c \leq 0 \leq c$.
   If $a > 0$, then $a = |a| \leq c$ and $-a < 0 < a = |a| \leq c$. So $-c \leq a$. Thus $-c \leq a \leq c$.
   If $a < 0$, then $-a = |a| \leq c$. So, $-c \leq a$ and $a < 0 < -a = |a| \leq c$. So $a \leq c$. Thus $-c \leq a \leq c$.
   $(\Leftarrow)$ Suppose $-c \leq a \leq c$. Then $a \leq c$ and $-c \leq a \Rightarrow -a \leq c$. Thus $a \leq c$ and $-a \leq c$. Hence $|a| \leq c$.

4. Using (iii) with $c = |a|$, we have $-|a| \leq a \leq |a|$.

***Theorem 1.9:*** ***[Triangle Inequality]***

If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.

***Proof:*** We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. By adding the two inequalites we get $-(|a|+|b|) \leq a+b \leq (|a|+|b|)$. Hence by the brevious theorem $|a + b| \leq |a| + |b|$.

***Theorem 1.10:*** *[]*

If $a, b \in \mathbb{R}$, then $||a| - |b|| \leq |a - b|$.

***Proof:*** Since $a = a - b + b$, then $|a| = |a - b + b| \leq |a - b| + |b|$, and hence $|a| - |b| \leq |a - b|$. Also, since $b = b - a + a$, then $|b| = |b - a + a| \leq |b - a| + |a|$, and hence $|b| - |a| \leq |b - a| = |a - b|$. Now, $|b| - |a| \leq |a - b| \Leftrightarrow -|a - b| \leq -|b| + |a|$ and hence $-|a - b| \leq |a| - |b|$. Also we have $|a| - |b| \leq |a - b|$. Thus $-|a - b| \leq |a| - |b| \leq |a| - |b| \leq |a - b|$. Therefore $||a| - |b|| \leq |a - b|$.