Subsequences and Bolzano-Weierstrass Theorem

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4.1 Subsequences and Bolzano-Weierstrass Theorem

Definition 4.1: Let $\{x_n\}_{n=1}^{\infty}$ be sequence of a real numbers and let $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a ***subsequence*** of $\{x_n\}_{n=1}^{\infty}$.

Example 4.1:

(a) If $\{x_n\} = \{\frac{1}{n}\}$ and if $n_k = 2k$ then $\{x_{n_k}\} = \{\frac{1}{2k}\}$ is a subsequence of $\{\frac{1}{n}\}$.

(b) $\{1 - \frac{1}{(2k-1)^2}\}$ is a subsequence of $\{1 - \frac{1}{n^2}\}$

(c) $\{1, 1, 1, \ldots\}$ is a subsequence of $\{(-1)^n\}$

Note 4.1: Let $\{x_n\}_{n=1}^{\infty}$ be sequence of a real numbers and let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Then $\lim\limits_{k\to\infty} x_{n_k} = x$. If for every $\epsilon > 0$ there exists $K \in \mathbb{N}$ such that if $k > K \Rightarrow |x_{n_k} - x| < \varepsilon$. Also note that $n_k \geq k \quad \forall k \in \mathbb{N}$.

Theorem 4.1: []

Let $\{x_n\}_{n=1}^{\infty}$ be sequence of a real numbers and suppose $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. Then any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ converges to x.

Proof: Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = x$, then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \epsilon$. Now, if $k > N \Rightarrow n_k \geq k > N \Rightarrow |x_{n_k} - x| < \epsilon$. Hence $\lim\limits_{k\to\infty} x_{n_k} = x$.

Note 4.2: Let $\{x_n\}_{n=1}^{\infty}$ be sequence of a real numbers. If $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that it is divergent, then $\{x_n\}_{n=1}^{\infty}$ diverges.

Definition 4.2: Let $\{x_n\}_{n=1}^{\infty}$ be sequence of an real numbers. We say that the term x_m is ***dominant "peak"*** if it is greater than every term which proceeds it: $x_k \leq x_m \quad \forall\, k \leq m$. Another way to say this is $x_m = \max\{x_k | k \leq m\}$.

Example 4.2:

(a) The sequence $\{1, 2, 3, \ldots, n, \ldots\}$ every term is a dominant.

(b) In the sequence $\{\frac{1}{n^2}\}$ has no dominant terms .

(c) In sequence $\{(-1)^n\}$ every even term is a peak.

(d) In sequence $\{1 - \frac{(-1)^n}{n}\}$ every odd term is a peak.

Theorem 4.2: **[Monotone Subsequence Theorem (MST):]**

Let $\{x_n\}_{n=1}^{\infty}$ be sequence of a real numbers, then $\{x_n\}_{n=1}^{\infty}$ has a monotone subsequence.

Proof: We will consider two cases.

Case1:

$\{x_n\}$ has infinitely many peaks terms. In this case we list the peaks by increasing indexes: $x_{n_1} \leq x_{n_2} \leq \cdots \leq x_{n_k} \leq \cdots$. Therefore the subsequence $\{x_{n_k}\}$ of peaks is a increasing subsequence of $\{x_n\}$.

Case2:

$\{x_n\}$ has a finite number of peaks terms. In this case we list the peaks by increasing indexes: $x_{m_1} \leq x_{m_2} \leq \cdots \leq x_{m_r}$. Let $n_1 = m_r$. Since x_{n_1} is the last peak then there exist x_{n_2} such that $x_{n_1} > x_{n_2}$. Since x_{n_2} is not a peak then there exists x_{n_3} such that $x_{n_2} > x_{n_3}$. Continuing in this way, we obtain an decreasing subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Theorem 4.3: **[The Bolzano-Weierstrass Theorem (BWT):]**

Let $\{x_n\}_{n=1}^{\infty}$ be bounded sequence of a real numbers, then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

Proof: By MST $\{x_n\}_{n=1}^{\infty}$ has a monotone subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Now, $\{x_{n_k}\}$ is monotone and bounded, then by MCT $\{x_{n_k}\}$ is convergent. Thus $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n\}$.