Series of Functions

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Definition 7.1: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and set $s_n = \sum_{k=1}^{n} f_k(x)$ for $x \in A$ and $n \geq 1$.

(i) We say that the series $\sum_{n=1}^{\infty} f_n(x)$ *converges pointwise on A* if $\{s_n(x)\}$ converges pointwise on A. [i.e. $\lim_{n\to\infty} s_n(x)$ exists for every $x \in A$.]

(ii) We say that the series $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on A* if $\{s_n(x)\}$ converges uniformly on A.

(iii) We say that the series $\sum_{n=1}^{\infty} f_n(x)$ *converges absolutely on A* if $\sum_{n=1}^{\infty} |f_n(x)|$ converges on A.

Example 7.1: Determine whether the given series of functions on the given interval converges or diverges

(a)

$$\sum_{n=0}^{\infty} x^n \quad x \in (-1, 1)$$

(b)

$$\sum_{n=0}^{\infty} x^n \quad x \in [0, \frac{1}{2}]$$

Solution:

(a)

The sequence of partial sum is $s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$ for $\forall\ x \in (-1, 1)$.Now, $\lim_{n\to\infty} s_n = \frac{1}{1 - x}$.

Hence $\sum_{n=0}^{\infty} x^n$ converges pointwise to $\frac{1}{1 - x}$ and the convergence is not uniform on $(-1, 1)$,

$$\text{since } \lim_{n\to\infty} \sup_{x\in(-1,1)} \left| s_n(x) - \frac{1}{1 - x} \right| = \lim_{n\to\infty} \sup_{x\in(-1,1)} \frac{|x|^{n+1}}{1 - x} = \infty.$$

(b)

The sequence of partial sum is $s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$ for $\forall\ x \in [0, \frac{1}{2}]$.Now, $\lim_{n\to\infty} s_n = \frac{1}{1 - x}$.

Hence $\sum_{n=0}^{\infty} x^n$ converges pointwise to $\frac{1}{1 - x}$ and the convergence is uniform on $[0, \frac{1}{2}]$,

$$\text{since } \lim_{n\to\infty} \sup_{x\in[0,\frac{1}{2}]} \left| s_n(x) - \frac{1}{1-x} \right| = \lim_{n\to\infty} \frac{(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = 0.$$

Theorem 7.1: ***The Cauchy Criterion For Uniform Convergence of a Series***

Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, defined on A. The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if for each $\epsilon > 0$, there is a number $N = N(\epsilon) \in \mathbb{N}$ such that if

$$m > n > N \Longrightarrow \left| \sum_{k=n+1}^{m} f_k(x) \right| < \epsilon.$$

Proof: The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if the sequence of partial sum $\{s_n(x)\}$ is Cauchy uniformly on A which is if and only if $\epsilon > 0$, there is a number $N = N(\epsilon) \in \mathbb{N}$ such that if

$$m > n > N \Longrightarrow \left| \sum_{k=n+1}^{m} f_k(x) \right| = |s_m(x) - s_n(x)| < \epsilon.$$

Corollary 7.1:

$$\text{If } \sum_{n=1}^{\infty} f_n \text{ converges uniformly on a set } A, \text{ then } \lim_{n\to\infty} \sup_{x\in A} |f_n(x)| = 0.$$

Proof:

Let $\epsilon > 0$ be given. Since $\sum_{n=1}^{\infty} f_n$ converges uniformly on a set A, there is $N \in \mathbb{N}$ such that if

$$m > n > N \Longrightarrow \left| \sum_{k=n+1}^{m} f_k(x) \right| < \frac{\epsilon}{2}.$$

$$\text{Now } n+1 > n > N \Longrightarrow |f_{n+1}(x)| = \left| \sum_{k=n+1}^{n+1} f_k(x) \right| < \frac{\epsilon}{2}.$$

$$\text{Thus, if, } n > N \Longrightarrow |f_{n+1}(x)| < \frac{\epsilon}{2} \Rightarrow \sup_{x\in A} |f_{n+1}(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

$$\text{Hence } \lim_{n\to\infty} \sup_{x\in A} |f_n(x)| = \lim_{n\to\infty} \sup_{x\in A} |f_{n+1}(x)| = 0.$$

Remark 7.1: If $\lim_{n\to\infty} \sup_{x\in A} |f_n(x)| \neq 0$, then the series $\sum_{n=1}^{\infty} f_n$ does not converges uniformly on A.

Example 7.2: Show that the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ does not converges uniformly on $(-2, 2)$.

Solution: Let $a \in (-2, 2)$. Now, $\sum_{n=1}^{\infty} \frac{a^n}{2^n} = \sum_{n=1}^{\infty} (\frac{a}{2})^n$ which is a convergent geometric series because $|r| = |\frac{a}{2}| < 1$. Hence the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ converges pointwise on $(-2,2)$. Now,

$$\sup_{x\in(-2,2)} \left|\frac{x^n}{2^n}\right| = 1 \text{ and hence } \lim_{n\to\infty} \sup_{x\in(-2,2)} \left|\frac{x^n}{2^n}\right| = 1 \neq 0.$$

Therefore by the above corollary the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ does not converges uniformly on $(-2,2)$.

Theorem 7.2: **Weierstrass's M-Test**

The series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if $|f_n(x)| \leq M_n$ for all $n \geq 1$ and for all $x \in A$ and the series of positive terms $\sum_{n=1}^{\infty} M_n$ converges.

Proof: The series $\sum_{n=1}^{\infty} M_n$ converges. Let $\epsilon > 0$, there is a number $N = N(\epsilon) \in \mathbb{N}$ such that if

$$m > n > N \Longrightarrow \sum_{k=n+1}^{m} M_k < \frac{\epsilon}{2}.$$

Now,

$$\text{if } m > n > N \Longrightarrow \left|\sum_{k=n+1}^{m} f_k(x)\right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k < \frac{\epsilon}{2} < \epsilon.$$

Hence the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

Example 7.3: Determine whether the given series of functions on the given interval converges pointwise , converges uniformly or, diverges

1. $$\sum_{n=0}^{\infty} n^2 e^{-nx} \quad x \in [1,\infty)$$

2. $$\sum_{n=0}^{\infty} \frac{1}{x^2 + 2^n} \quad x \in \mathbb{R}$$

3. $$\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2} \quad x \in \mathbb{R}$$

4. $$\sum_{n=0}^{\infty} \left(\frac{1}{n}\right)^{-x} \quad x \in [1,\infty)$$

Solution:

1.

Since $|n^2e^{-nx}| \leq n^2e^{-n}$ (show that) $\forall x \in [1,\infty)$, and $\forall n \geq 1$, and since $\sum_{n=0}^{\infty} n^2e^{-n}$ converges by Ratio test (show that), then $\sum_{n=0}^{\infty} n^2e^{-nx}$ converges uniformly on $[1,\infty)$ by Weierstrass's M-Test.

2.

Since $x^2+2^n \geq 2^n \Rightarrow \dfrac{1}{x^2+2^n} \leq \dfrac{1}{2^n}$ $\forall x \in \mathbb{R}$, and $\forall n \geq 1$, and since $\sum_{n=0}^{\infty} \dfrac{1}{2^n}$ converges by Geometric series test (show that), then $\sum_{n=0}^{\infty} \dfrac{1}{x^2+2^n}$ converges uniformly on $\mathbb{R}$ by Weierstrass's M-Test.

3.

Since $\left|\dfrac{\sin(nx)}{n^2}\right| \leq \dfrac{1}{n^2}$ $\forall x \in \mathbb{R}$, and $\forall n \geq 1$, and since $\sum_{n=0}^{\infty} \dfrac{1}{n^2}$ converges by $p-$ series test (show that), then $\sum_{n=0}^{\infty} \dfrac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$ by Weierstrass's M-Test.

4.

Since $\left(\dfrac{1}{n}\right)^{-x} = n^x \geq n$ $\forall x \in [1,\infty)$, and $\forall n \geq 1$ and since $\lim_{n\to\infty} n = \infty$, then $\lim_{n\to\infty} \left(\dfrac{1}{n}\right)^{-x} = \infty$. Hence by Divergence test the series $\sum_{n=0}^{\infty} \left(\dfrac{1}{n}\right)^{-x}$ $x \in [1,\infty)$ diverges .

Theorem 7.3: []

If the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A and if each $f_n(x)$ is continuous on A, then $\sum_{n=1}^{\infty} f_n(x)$ is continuous on A.

If the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ and if each $f_n(x)$ is integrable on $[a,b]$, then $\sum_{n=1}^{\infty} f_n(x)$ is integrable on $[a,b]$ and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Proof: The proof is left as an exercises for you!!!

Example 7.4:

Let $f(x) = \sum_{n=0}^{\infty} \left(\dfrac{1}{1+x}\right)^n$ $x \in [1,\infty)$. Show that f is continuous on $[1,\infty)$.

Solution:

$$\text{Since } x+1 \geq 2 \Rightarrow \frac{1}{x+1} \leq \frac{1}{2} \Rightarrow \left(\frac{1}{x+1}\right)^n \leq \left(\frac{1}{2}\right)^n \quad \forall x \in [1,\infty),$$

and $\forall\, n \geq 1$, and since $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$ converges by Geometric series test (show that), then

$\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{1+x}\right)^n$ converges uniformly on $[1, \infty)$ by Weierstrass's M-Test.

Now, since each $\left(\dfrac{x}{1+x}\right)^n$ is continuous , then $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{1+x}\right)^n$ is continuous on $[1, \infty)$.

Example 7.5:

$$\text{Prove that } \int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\, dx = \frac{e}{e^2-1}.$$

Solution:

Since $|n^2 e^{-nx}| \leq n^2 e^{-n}$ (show that) $\quad \forall\, x \in [1, \infty)$, and $\forall\, n \geq 1$, and since $\displaystyle\sum_{n=0}^{\infty} n^2 e^{-n}$ converges by

Ratio test (show that), then $\displaystyle\sum_{n=0}^{\infty} n^2 e^{-nx}$ converges uniformly on $[1, \infty)$ by Weierstrass's M-Test.

Hence $\displaystyle\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\, dx = \sum_{n=1}^{\infty} \int_1^2 ne^{-nx}\, dx = \sum_{n=1}^{\infty} \left[-e^{-nx}\right]_1^2 = \sum_{n=1}^{\infty} \left[-e^{-2n} + e^{-n}\right] = -\sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$

Thus $\displaystyle\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\, dx = -\sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{-\frac{1}{e^2}}{1-\frac{1}{e^2}} + \frac{\frac{1}{e}}{1-\frac{1}{e}} = \frac{-1}{e^2-1} + \frac{1}{e-1} = \frac{-1}{e^2-1} + \frac{e+1}{e^2-1} = \frac{e}{e^2-1}$
