Sequence of Functions

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Definition 6.1: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and let $f : A \to \mathbb{R}$ be a function. We say that $\{f_n(x)\}_{n=1}^{\infty}$ *converges pointwise to* $f(x)$ if for each $x \in A$

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and we write

$$f_n(x) \xrightarrow{p.w.} f(x)$$

Example 6.1: Let $f_n : (-1, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Find $f(x) = \lim_{n\to\infty} f_n(x)$.

Solution:

$$\text{For } x = 1, \Rightarrow f_n(1) = (1)^n = 1 \text{ and } \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1.$$

$$\text{For } -1 < x < 1, \Rightarrow \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0.$$

$$\text{Hence } f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{for } -1 < x < 1; \\ 1, & \text{for } x = 1. \end{cases}$$

Example 6.2: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x}{n}$. Find $f(x) = \lim_{n\to\infty} f_n(x)$.

Solution: For any $x \in \mathbb{R}$, we have $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{x}{n} = x \lim_{n\to\infty} \dfrac{1}{n} = x \cdot 0 = 0.$

Example 6.3: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x + nx^2}{n}$. Find $f(x) = \lim_{n\to\infty} f_n(x)$.

Solution: For any $x \in \mathbb{R}$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x + nx^2}{n} = \lim_{n\to\infty} \left[\frac{x}{n} + \frac{nx^2}{n}\right] = x \lim_{n\to\infty} \frac{1}{n} + x^2 \lim_{n\to\infty} 1 = 0 + x^2 = x^2.$$

Example 6.4: Let $f_n : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x^n}{1+x^n}$. Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

Solution: For any $-1 < x < 1$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{\lim_{n\to\infty} x^n}{\lim_{n\to\infty}(1+x^n)} = \frac{0}{1+0} = 0.$$

$$\text{For } x = 1, \text{ we have } f_n(1) = \frac{1^n}{1+1^n} = \frac{1}{2}, \text{ hence } f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

$$\text{For } |x| > 1, \text{ we have } f_n(x) = \frac{x^n}{1+x^n} = \frac{1}{(\frac{1}{x})^n + 1}, \text{ hence } f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{(\frac{1}{x})^n+1} = \frac{1}{0+1} = 1.$$

Hence

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{if } -1 < x < 1; \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}; \\ 1, & \text{if } |x| > 1. \end{cases}$$

Remark 6.1: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and let $f : A \to \mathbb{R}$ be a function. Then $\{f_n(x)\}_{n=1}^{\infty}$ *converges pointwise to* $f(x)$ if and only if for each $x \in A$ and for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ ($N = N(x, \epsilon)$ i.e. N depend on x and ϵ) such that

$$\text{if } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

Example 6.5: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x^2 + nx}{n}$.

(a) Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

(b) Prove your answer in part (**a**) using the definition

Solution: (a) For any $x \in \mathbb{R}$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left[\frac{x^2}{n} + \frac{nx}{n}\right] = \lim_{n\to\infty} \left[\frac{x^2}{n} + x\right] = x.$$

(b)

$$\text{Note that for } x \neq 0, \text{ we have } |f_n(x) - f(x)| = \left|\frac{x^2}{n} + x - x\right| = \frac{|x|^2}{n} \text{ and for } x = 0, \text{ we have } |f_n(0) - f(0)| = 0.$$

Hence if we let $\dfrac{|x|^2}{n} < \epsilon \Rightarrow \dfrac{n}{|x|^2} > \dfrac{1}{\epsilon} \Rightarrow n > \dfrac{|x|^2}{\epsilon}$.

To prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$. Let $\epsilon > 0$ be given and let $x \in \mathbb{R}$. Choose $N \in \mathbb{N}$ such that $N \geq \dfrac{|x|^2}{\epsilon}$.

$$\text{If } n > N \Rightarrow \frac{1}{n} < \frac{1}{N} \leq \frac{\epsilon}{|x|^2}.$$
$$n > N \Rightarrow \frac{1}{n} < \frac{\epsilon}{|x|^2}.$$
$$n > N \Rightarrow \frac{|x|^2}{n} < \epsilon.$$
$$\text{Now, if } n > N \Rightarrow |f_n(x) - f(x)| = \left|\frac{x^2}{n} + x - x\right| = \frac{|x|^2}{n} < \epsilon.$$
$$\text{Now, if } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$
$$\text{Therefore } \lim_{n\to\infty} f_n(x) = f(x).$$

Definition 6.2: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and let $f : A \to \mathbb{R}$ be a function. We say that $\{f_n(x)\}_{n=1}^{\infty}$ *converges uniformly to* $f(x)$,and we write $f_n(x) \xrightarrow{U.} f(x)$ if $\epsilon > 0$ there exist $N \in \mathbb{N}$ ($N = N(\epsilon)$ i.e. N depend on ϵ only) such that

$$\text{if } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon \text{ for every } x \in A.$$

Example 6.6: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{nx+1}{n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, prove your answer in part (**a**) using the definition.

Solution: (a) For any $x \in \mathbb{R}$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\frac{nx}{n} + \frac{1}{n}\right] = \lim_{n\to\infty}\left[x + \frac{1}{n}\right] = x.$$

(b)

$$\text{Note that for any } x, \text{ we have } |f_n(x) - f(x)| = \left|x + \frac{1}{n} - x\right| = \frac{1}{n}$$

Hence if we let $\dfrac{1}{n} < \epsilon \Rightarrow n > \dfrac{1}{\epsilon}$.

To prove that $\lim_{n\to\infty} f_n(x) = f(x)$ uniformly. Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ such that $N \geq \dfrac{1}{\epsilon}$.

$$\text{If } n > N \Rightarrow \frac{1}{n} < \frac{1}{N} \leq \epsilon.$$
$$\text{Now, if } n > N \Rightarrow |f_n(x) - f(x)| = \left|\frac{1}{n} + x - x\right| = \frac{1}{n} < \epsilon.$$
$$\text{Now, if } n > N \Rightarrow |f_n(x) - f(x)| < \epsilon.$$
$$\text{Therefore } \lim_{n\to\infty} f_n(x) = f(x) \text{ uniformly.}$$

Lemma 6.1: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and for each $x \in A$ let $f(x) = \lim_{n\to\infty} f_n(x)$. If $x_0 \in A$ and $\{x_n\} \subset A$ sequence such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f_n(x_n) \neq f(x_0)$, then $\{f_n(x)\}$ does not converge uniformly on A.

Proof: This is left as an exercise.

Example 6.7: Let $f_n : [1, \infty) \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x^n}{x^n + 1}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, prove your answer in part (**a**) .

Solution: (a)

$$\text{For } x = 1, \text{ we have } f(1) = \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} \frac{1^n}{1^n + 1} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}.$$

$$\text{For } x > 1, \text{ we have } f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{x^n + 1} = \lim_{n\to\infty} \frac{1}{1 + (\frac{1}{x})^n} = \frac{1}{1+0} = 1.$$

Thus

$$f(x) = \begin{cases} \frac{1}{2}, & \text{for } x = 1; \\ 1, & \text{for } x > 1. \end{cases}$$

(b) Now, let $x_n = 1 + \dfrac{1}{n} \in [0, \infty)$ and $\lim_{n\to\infty} (1 + \dfrac{1}{n}) = 1 \in [0, \infty)$,

but $\lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} f_n(\left(1 + \dfrac{1}{n}\right)) = \lim_{n\to\infty} \dfrac{\left(1+\frac{1}{n}\right)^n}{1 + \left(1 + \frac{1}{n}\right)^n} = \dfrac{e}{1+e} \neq \dfrac{1}{2} = f(1)$. Hence $\{f_n(x)\}$ does not converge uniformly on $[1, \infty)$.

Theorem 6.1: []

Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and for each $x \in A$ let $f(x) = \lim_{n\to\infty} f_n(x)$.

$$\text{Then } \{f_n\} \text{ converges uniformly to } f \text{ on } A \text{ if and only if } \lim_{n\to\infty} \left[\sup_{x\in A} |f_n(x) - f(x)|\right] = 0.$$

Proof: ($\Rightarrow$) Suppose that $\{f_n\}$ converges uniformly to f on A and let $\epsilon > 0$ be given.

$$\text{Since } f_n(x) \xrightarrow{U.} f(x), \text{ then there exist } N \in \mathbb{N} \text{ such that}$$

$$\text{if } n > N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \forall\, x \in A.$$

$$\text{Hence if } n > N \Rightarrow \sup_{x\in A} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

$$\text{Thus } \lim_{n\to\infty} \left[\sup_{x\in A} |f_n(x) - f(x)|\right] = 0.$$

($\Leftarrow$) Suppose that $\lim_{n\to\infty}\left[\sup_{x\in A}|f_n(x)-f(x)|\right]=0$ and let $\epsilon>0$ be given.

$$\text{Since } \lim_{n\to\infty}\left[\sup_{x\in A}|f_n(x)-f(x)|\right]=0, \text{ then there exist } N\in\mathbb{N} \text{ such that}$$

$$\text{if } n>N \Rightarrow \sup_{x\in A}|f_n(x)-f(x)|<\epsilon \quad \forall\, x\in A.$$

$$\text{Hence if } n>N \Rightarrow |f_n(x)-f(x)|\le \sup_{x\in A}|f_n(x)-f(x)|<\epsilon \quad \forall\, x\in A.$$

$$\text{Thus } \lim_{n\to\infty} f_n(x)=f(x) \text{ uniformly on } A.$$

Example 6.8: Let $f_n:[0,1]\to\mathbb{R}$ defined by $f_n(x)=x^n(1-x)$.

(a) Find $f(x)=\lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x)\xrightarrow{U.} f(x)$, prove your answer.

Solution:

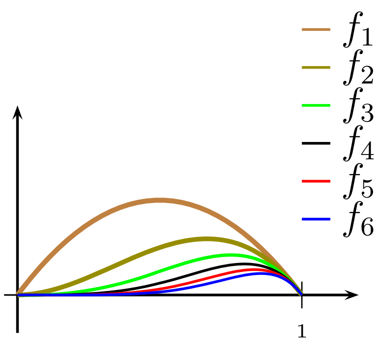

Figure 1:

(a) For any $x\in[0,1)$, we have

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}[x^n(1-x)]=(1-x)\lim_{n\to\infty}x^n=(1-x)\cdot 0=0 \text{ and } f(1)=\lim_{n\to\infty} f_n(1)=0.$$

Hence $f(x)=0$ for every $x\in[0,1]$.

(b)To show that $f_n(x)\xrightarrow{U.} f(x)$ we need to prove that $\lim_{n\to\infty}\left[\sup_{x\in[0,1]}|f_n(x)-f(x)|\right]=0$. We can use calculus to find the sup for each $f_n(x)-f(x)$.

For any $x\in[0,1]$, let $g_n(x)=f_n(x)-f(x)=x^n(1-x)-0=x^n(1-x)$. Thus , $g_n'(x)=nx^{n-1}(1-x)-x^n$.

Hence $g_n'(x)=[n(1-x)-x]x^{n-1}=[n-(n+1)x]x^{n-1}$. Therefore $g_n'(x)=0\Rightarrow x=0$ or $x=\dfrac{n}{n+1}\in(0,1)$.

Now, since $g_n(0) = 0 = g_n(1)$, then $g_n(x)$ may have maximum at $x = \dfrac{n}{n+1}$. We use the first derivative test for that, and we get

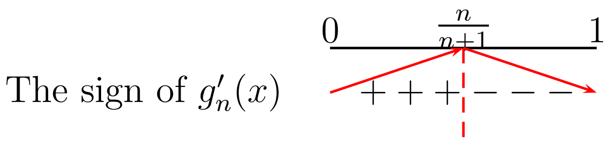

Figure 2: The sign of $g_n'(x)$

Hence $g_n(x)$ has a maximum at $x = \dfrac{n}{n+1}$ and its value is $g_n\left(\dfrac{n}{n+1}\right) = \left(\dfrac{n}{n+1}\right)^n \left(1 - \dfrac{n}{n+1}\right)$.
Hence $g_n\left(\dfrac{n}{n+1}\right) = \left(1 - \dfrac{1}{n+1}\right)^n \left(1 - \dfrac{n}{n+1}\right)$.

$$\text{Now, } \lim_{n\to\infty}\left[\sup_{x\in[0,1]} |f_n(x) - f(x)|\right] = \lim_{n\to\infty} g_n\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left[\left(1 - \frac{1}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)\right] = e^{-1}\cdot 0 = 0.$$

Hence $f_n(x) \xrightarrow{U.} 0$.

Example 6.9: Let $f_n : (-1,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, prove your answer.

Solution:

We have $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{for } -1 < x < 1; \\ 1, & \text{for } x = 1. \end{cases}$

Hence

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup \begin{cases} x^n, & \text{for } 0 \le x < 1; \\ 0, & \text{for } x = 1. \end{cases} = 1$$

Thus

$$\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Hence $f_n(x) \overset{U.}{\not\longrightarrow} f(x)$

Example 6.10: Let $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = nx^n(1-x)$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, prove your answer.

Solution:

(a) For any $x \in [0,1)$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}[nx^n(1-x)] = (1-x)\lim_{n\to\infty} nx^n = (1-x)\cdot 0 = 0 \text{ and } f(1) = \lim_{n\to\infty} f_n(1) = 0.$$

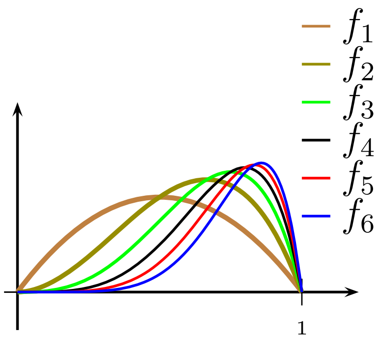

Figure 3:

Hence $f(x) = 0$ for every $x \in [0,1]$.

(b)To show that $f_n(x) \xrightarrow{U.} f(x)$ we need to prove that $\lim\limits_{n\to\infty}\left[\sup\limits_{x\in[0,1]}|f_n(x) - f(x)|\right] = 0$. We can use calculus to find the sup for each $f_n(x) - f(x)$.

For any $x \in [0,1]$, let $g_n(x) = f_n(x) - f(x) = nx^n(1-x) - 0 = nx^n(1-x)$. Thus , $g_n'(x) = n^2x^{n-1}(1-x) - nx^n$.

Hence $g_n'(x) = [n^2(1-x) - nx]x^{n-1} = [n^2 - (n^2+n)x]x^{n-1}$. Therefore $g_n'(x) = 0 \Rightarrow x = 0$ or $x = \dfrac{n}{n+1} \in (0,1)$. Now, since $g_n(0) = 0 = g_n(1)$, then $g_n(x)$ may have maximum at $x = \dfrac{n}{n+1}$. We use the first derivative test for that, and we get

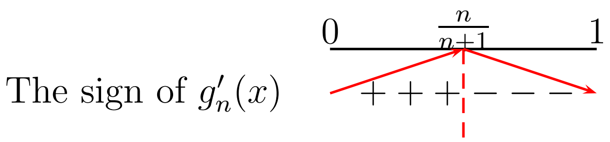

Figure 4: The sign of $g_n'(x)$

Hence $g_n(x)$ has a maximum at $x = \dfrac{n}{n+1}$ and its value is $g_n\left(\dfrac{n}{n+1}\right) = n\left(\dfrac{n}{n+1}\right)^n\left(1 - \dfrac{n}{n+1}\right)$.

Hence $g_n\left(\dfrac{n}{n+1}\right) = \left(1 - \dfrac{1}{n+1}\right)^n\left(n - \dfrac{n^2}{n+1}\right) = \left(1 - \dfrac{1}{n+1}\right)^n\left(\dfrac{n}{n+1}\right)$.

$$\text{Now, } \lim_{n\to\infty}\left[\sup_{x\in[0,1]}|f_n(x) - f(x)|\right] = \lim_{n\to\infty} g_n\left(\frac{n}{n+1}\right) = \lim_{n\to\infty}\left[\left(1 - \frac{1}{n+1}\right)^n\left(\frac{n}{n+1}\right)\right] = e^{-1}\cdot 1 = e^{-1} \neq 0.$$

Hence $f_n(x) \overset{U.}{\not\longrightarrow} 0$.

Theorem 6.2: []

Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of continuous functions $n \geq 1$, and for each $x \in A$ let $f(x) = \lim\limits_{n\to\infty} f_n(x)$. If $\{f_n\}$ converges uniformly to f on A, then $f(x)$ is continuous on A.

Proof: We will show that f is continuous at any point $x_0 \in A$. Let $\epsilon > 0$ be given, since $f_n(x) \xrightarrow{U.} f(x)$, then there exist $N \in \mathbb{N}$ such that if $n > N \Longrightarrow |f_n(x) - f(x)| < \dfrac{\epsilon}{3} \quad \forall\, x \in A$.

Now, $N+1 > N \Longrightarrow |f_{N+1}(x) - f(x)| < \dfrac{\epsilon}{3} \quad \forall\, x \in A$.

Since f_{N+1} is continuous on A, then f_{N+1} is continuous at x_0. Then there exist $\delta > 0$ such that if $|x - x_0| < \delta \Longrightarrow |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\epsilon}{3}$.

$$\begin{aligned}\text{Now, if } |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| &= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(x_0) + f_{N+1}(x_0) - f(x_0)| \\ &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Example 6.11: Let $f_n : (-1, 1] \to \mathbb{R}$ defined by $f_n(x) = x^n$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $(-1, 1]$ prove your answer.

Solution:

We have $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{for } -1 < x < 1; \\ 1, & \text{for } x = 1. \end{cases}$ Now, since $f_n(x) = x^n$ is continuous on $[0, 1]$ for each $n \in \mathbb{N}$, and $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & \text{for } -1 < x < 1; \\ 1, & \text{for } x = 1. \end{cases}$ is not continuous at $x = 1$, then by the above theorem $f_n(x) \not\xrightarrow{U.} f(x)$ on $[0, 1]$.

Example 6.12: Let $f_n : (0, \infty) \to \mathbb{R}$ defined by $f_n(x) = ne^{-nx}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $(0, \infty)$? prove your answer.

(c) Does $f_n(x) \xrightarrow{U.} f(x)$, on $[a, \infty)$? $\quad a > 0$, prove your answer.

Solution:

(a) We have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} ne^{-nx} = \lim_{n\to\infty} \frac{n}{(e^x)^n} = 0.$$

(b) Now,

$$\lim_{n\to\infty} \left[\sup_{x\in(0,\infty)} |f_n(x) - f(x)| \right] = \lim_{n\to\infty} \left[\sup_{x\in(0,\infty)} \frac{n}{e^{nx}} \right] = \lim_{n\to\infty} n = \infty \neq 0,$$

then $f_n(x) \not\xrightarrow{U.} f(x)$ on $(0, \infty)$.

(c) Since

$$\lim_{n\to\infty} \left[\sup_{x\in[a,\infty)} |f_n(x) - f(x)| \right] = \lim_{n\to\infty} \left[\sup_{x\in[a,\infty)} \frac{n}{e^{nx}} \right] = \lim_{n\to\infty} \frac{n}{e^{an}} = 0,$$

then $f_n(x) \xrightarrow{U.} f(x)$ on $[a, \infty)$.

Example 6.13: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $\mathbb{R}$? prove your answer.

(c) Does $f_n(x) \xrightarrow{U.} f(x)$, on $[-a, a]$? $a > 0$, prove your answer.

Solution:

(a) We have $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n} = 0$.

(b) Now,

$$\lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|\right] = \lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} \frac{|x|}{n}\right] = \infty \neq 0,$$

then $f_n(x) \not\xrightarrow{U.} f(x)$ on $\mathbb{R}$.

(c) Since

$$\lim_{n\to\infty}\left[\sup_{x\in[-a,a]} |f_n(x) - f(x)|\right] = \lim_{n\to\infty}\left[\sup_{x\in[-a,a]} \frac{|x|}{n}\right] = \lim_{n\to\infty} \frac{a}{n} = 0,$$

then $f_n(x) \xrightarrow{U.} f(x)$ on $[-a, a]$.

Definition 6.3: Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$. We say that $\{f_n(x)\}_{n=1}^{\infty}$ *is uniformly Cauchy* on A if for each $\epsilon > 0$, there ia a number $N = N(\epsilon) \in \mathbb{N}$ such that if $n, m > N \Longrightarrow |f_n(x) - f_m(x)| < \epsilon$.

Theorem 6.3: ***The Cauchy Criterion For Uniform Convergence***

Let A be a nonempty set of real numbers. Let $f_n : A \to \mathbb{R}$ be a sequence of functions $n \geq 1$, and for each $x \in A$ let $f(x) = \lim_{n\to\infty} f_n(x)$. Then $\{f_n\}$ converges uniformly to f on A if and only if $\{f_n\}$ uniformly Cauchy on A. ***Proof:***
($\Rightarrow$) Suppose that $\{f_n\}$ converges uniformly to f on A and let $\epsilon > 0$ be given.

$$\begin{aligned}
&\text{Since } f_n(x) \xrightarrow{U.} f(x), \text{ then there exist } N \in \mathbb{N} \text{ such that}\\
&\text{if } n > N \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \forall\, x \in A.\\
&\text{Hence if } m > N \Rightarrow |f_m(x) - f(x)| < \frac{\epsilon}{2} \quad \forall\, x \in A.\\
&\text{Thus if } n, m > N \Rightarrow |f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|\\
&\qquad \leq |f_n(x) - f(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\\
&\text{Thus if } n, m > N \Rightarrow |f_n(x) - f_m(x)| < \epsilon \quad \forall\, x \in A.
\end{aligned}$$

Therefore $\{f_n\}$ is uniformly Cauchy on A.

($\Leftarrow$) Suppose that $\{f_n\}$ is uniformly Cauchy on A. and let $\epsilon > 0$ be given.

Since $\{f_n\}$ is uniformly Cauchy on A, then there exist $N \in \mathbb{N}$ such that

$$\text{if } n, m > N \Rightarrow |f_n(x) - f_m(x)| < \frac{\epsilon}{2} \quad \forall\, x \in A. ------(1)$$

Hence for a fixed $x \in A\{f_n(x)\}$ is Cauchy sequence $\Rightarrow f(x) = \lim_{n\to\infty} f_n(x)$ exist.

Now, let $m \to \infty$ in (1), we get, if $n > N \Rightarrow |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon \quad \forall\, x \in A$.

Then $\{f_n\}$ converges uniformly to f on A.

Theorem 6.4: []

Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of integrable functions $n \geq 1$, and for each $x \in A$ let $f(x) = \lim_{n\to\infty} f_n(x)$. Suppose $\{f_n\}$ converges uniformly to f on $[a, b]$, then $f(x)$ is integrable on $[a, b]$ and

$$\int_a^b f(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

Proof: Let $\epsilon > 0$ be given, since $f_n(x) \xrightarrow{U.} f(x)$, then there exist $N \in \mathbb{N}$ such that
if $n > N \Longrightarrow |f_n(x) - f(x)| < \dfrac{\epsilon}{3(b-a)} \qquad \forall\, x \in [a, b]$.
Now, $N + 1 > N \Longrightarrow |f_{N+1}(x) - f(x)| < \dfrac{\epsilon}{3(b-a)} \qquad \forall\, x \in [a, b]$.
Hence $-\dfrac{\epsilon}{3(b-a)} < f_{N+1}(x) - f(x) < \dfrac{\epsilon}{3(b-a)} \qquad \forall\, x \in [a, b]$.
Since f_{N+1} is integrable on $[a, b]$, then there exist a partition $P = \{a = x_0 < x_1 < ... < x_{n-1} < x_n = b\}$ of $[a, b]$ such that $U(f_{N+1}, P) - L(f_{N+1}, P) < \dfrac{\epsilon}{3}$.

Note that

$$\sup_{x\in[x_{k-1},x_k]} f(x) = \sup_{x\in[x_{k-1},x_k]} [f(x) - f_{N+1}(x) + f_{N+1}(x)] \leq \sup_{x\in[x_{k-1},x_k]} [f(x) - f_{N+1}(x)] + \sup_{x\in[x_{k-1},x_k]} f_{N+1}(x) \quad (*)$$

and

$$\inf_{x\in[x_{k-1},x_k]}[f(x) - f_{N+1}(x)] + \inf_{x\in[x_{k-1},x_k]} f_{N+1}(x) \leq \inf_{x\in[x_{k-1},x_k]}[f(x) - f_{N+1}(x) + f_{N+1}(x)] = \inf_{x\in[x_{k-1},x_k]} f(x) \quad (**).$$

$$\begin{aligned}
\text{Now, } U(f,P) &= \sum_{k=1}^{n} M_k(f)\triangle x_k \\
&= \sum_{k=1}^{n} M_k(f - f_{N+1} + f_{N+1})\triangle x_k \\
&\leq \sum_{k=1}^{n} M_k(f - f_{N+1})\triangle x_k + \sum_{k=1}^{n} M_k(f_{N+1})\triangle x_k \quad \text{using } (*) \\
&< \sum_{k=1}^{n} \frac{\epsilon}{3(b-a)}\triangle x_k + \sum_{k=1}^{n} M_k(f_{N+1})\triangle x_k \\
&= \frac{\epsilon}{3(b-a)} \sum_{k=1}^{n} \triangle x_k + U(f_{N+1},P) \\
&= \frac{\epsilon}{3(b-a)}(b-a) + U(f_{N+1},P) \\
\text{Thus } U(f,P) &< \frac{\epsilon}{3} + U(f_{N+1},P) \\
\text{Similarly, one can show } -L(f,P) &< \frac{\epsilon}{3} - L(f_{N+1},P) \\
\text{Hence } U(f,P) - L(f,P) &< 2\frac{\epsilon}{3} + U(f_{N+1},P) - L(f_{N+1},P) < 2\frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Hence f is integrable on $[a,b]$.

Since $f_n(x) \xrightarrow{U.} f(x)$, then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |f_n(x) - f(x)| < \dfrac{\epsilon}{3(b-a)} \quad \forall\, x \in [a,b]$.

$$\text{Now, if } n > N \Rightarrow \left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| \leq \int_a^b |f_n(x) - f(x)|\, dx < \int_a^b \frac{\epsilon}{3(b-a)}\, dx = \frac{\epsilon}{3(b-a)}(b-a) < \epsilon.$$

$$\text{Thus } \lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx.$$

Example 6.14: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{nx+1}{n+nx^2}$.

(a) Find $f(x) = \lim\limits_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $\mathbb{R}$? prove your answer.

(c) Evaluate $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)\, dx$.

Solution:

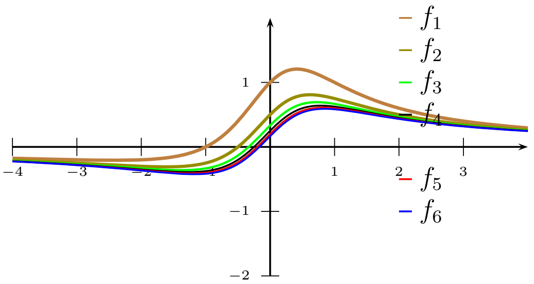

Figure 5:

(a) Note that $f_n(x) = \dfrac{nx+1}{n+nx^2} = \dfrac{x}{1+x^2} + \dfrac{1}{n(1+x^2)}$. Hence

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\frac{x}{1+x^2} + \frac{1}{n(1+x^2)}\right] = \frac{x}{1+x^2}.$$

(b) Now,

$$\lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|\right] = \lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} \frac{1}{n(1+x^2)}\right] = \lim_{n\to\infty}\frac{1}{n} = 0,$$

then $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$.
(c) Since $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$, then $f_n(x) \xrightarrow{U.} f(x)$ on $[0,1]$. Hence

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\ln(1+x^2)\right]_0^1 = \frac{1}{2}\ln 2 = \ln\sqrt{2}.$$

Example 6.15: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{x+n}{n+nx^2}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $\mathbb{R}$? prove your answer.

(c) Evaluate $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$.

Solution:

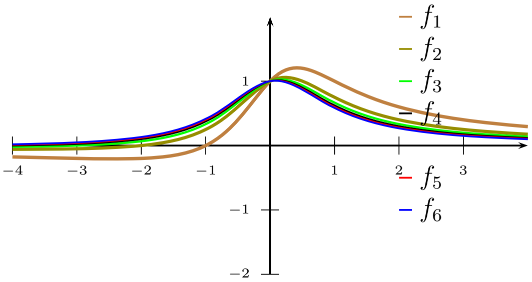

Figure 6:

(a) Note that $f_n(x) = \dfrac{x+n}{n+nx^2} = \dfrac{\frac{x}{n}}{1+x^2} + \dfrac{1}{1+x^2}$. Hence

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\frac{\frac{x}{n}}{1+x^2} + \frac{1}{1+x^2}\right] = \frac{1}{1+x^2}.$$

(b) Now, let $g_n(x) = f_n(x) - f(x) = \dfrac{\frac{x}{n}}{1+x^2} + \dfrac{1}{1+x^2} - \dfrac{1}{1+x^2} = \dfrac{x}{n(1+x^2)}$.

$$g'_n(x) = \frac{n + nx^2 - x(2nx)}{n^2(1+x^2)^2} = \frac{n - nx^2}{n^2(1+x^2)^2} = \frac{1-x^2}{n(1+x^2)}.$$

$g'_n(x) = 0 \Longleftrightarrow 1 - x^2 = 0 \Longleftrightarrow x = \pm 1.$

The sign of $g'_n(x)$

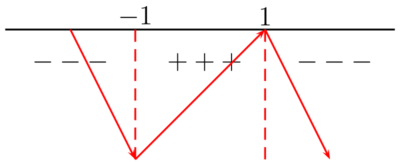

Figure 7:

Hence $|g_n(x)|$ has a maximum at $x = \pm 1$.

$$\lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|\right] = \lim_{n\to\infty}\left[\sup_{x\in\mathbb{R}} \frac{|x|}{n(1+x^2)}\right] = \lim_{n\to\infty} \frac{1}{2n} = 0,$$

then $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$.

(c) Since $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$, and each $f_n(x)$ is continuous, then $f_n(x) \xrightarrow{U.} f(x)$ on $[0,1]$ and each $f_n(x)$ is integrable. Hence

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Example 6.16: Let $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \dfrac{n + \sin x}{2n + \cos^2 x}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$.

(b) Does $f_n(x) \xrightarrow{U.} f(x)$, on $\mathbb{R}$? prove your answer.

(c) Evaluate $\lim_{n\to\infty} \int_1^5 f_n(x)\,dx$.

(d) Evaluate $\lim_{n\to\infty} \lim_{x\to(2+\frac{\pi}{3})} f_n(x)$.

Solution:

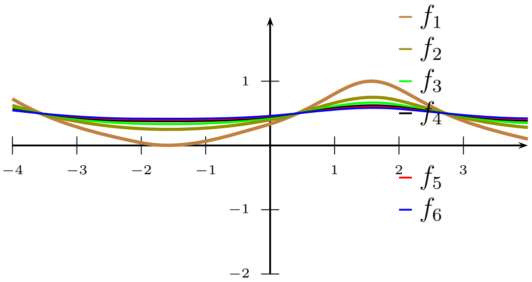

Figure 8:

(a)

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\frac{n+\sin x}{2n+\cos^2 x}\right] = \lim_{n\to\infty}\left[\frac{1+\frac{\sin x}{n}}{2+\frac{\cos^2 x}{n}}\right] = \frac{1}{2}.$$

(b) Now, let $f_n(x) - f(x) = \dfrac{n + \sin x}{2n + \cos^2 x} - \dfrac{1}{2} = \dfrac{2n + \sin x - 2n - \cos^2 x}{2(2n + \cos^2 x)} = \dfrac{\sin x - \cos^2 x}{2(2n + \cos^2 x)}$.

Hence $|f_n(x) - f(x)| = \left|\dfrac{\sin x - \cos^2 x}{2(2n + \cos^2 x)}\right| \le \dfrac{|\sin x| + |\cos^2 x|}{2(2n + \cos^2 x)} \le \dfrac{2}{4n} = \dfrac{1}{2n} \qquad \forall\, x \in \mathbb{R}$.

$$\text{Thus } |f_n(x) - f(x)| \le \frac{1}{2n} \qquad \forall\, x \in \mathbb{R}, \text{ and hence } 0 \le \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \le \frac{1}{2n}.$$

$$\text{Since } 0 \le \lim_{n\to\infty} \left[\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|\right] \le \lim_{n\to\infty} \frac{1}{2n} = 0, \text{ then } \lim_{n\to\infty} \left[\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|\right] = 0.$$

Therefore $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$.

(c) Since $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$, and each $f_n(x)$ is continuous, then $f_n(x) \xrightarrow{U.} f(x)$ on $[0, 1]$ and each $f_n(x)$ is integrable. Hence

$$\lim_{n\to\infty} \int_1^5 f_n(x)\, dx = \int_1^5 \lim_{n\to\infty} f_n(x)\, dx = \int_1^5 f(x)\, dx = \int_1^5 \frac{1}{2}\, dx = \frac{1}{2}(5 - 1) = 2.$$

(d) Since $f_n(x) \xrightarrow{U.} f(x)$ on $\mathbb{R}$, and each $f_n(x)$ is continuous, then $f(x)$ is continuous and

$$\lim_{n\to\infty} \lim_{x\to(2+\frac{\pi}{3})} f_n(x) = \lim_{x\to(2+\frac{\pi}{3})} \lim_{n\to\infty} f_n(x) = \lim_{x\to(2+\frac{\pi}{3})} \frac{1}{2} = \frac{1}{2}.$$

