Sequences and Their Limits

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1.1 Sequence

Definition 1.1: A sequence of a real numbers is a real-valued function whose domain is the set of natural numbers $\mathbb{N}$, or a subset of it.

Note 1.1: We use a symbol x_n to represent the range of the function and $\{x_n\}_{n=1}^{\infty}$, or $\{x_n\}$ to represent the sequence itself. The following notation to indicate a sequence:

$$\{x_1, x_2, x_3, x_4, \dots, x_n, \dots\}\ , \quad or \quad \{x_n\}_{n=1}^{\infty}$$

$$\underset{\text{First term}}{\uparrow} \qquad\qquad \underset{n^{\text{th}}\text{ term}}{\uparrow}$$

Example 1.1:

(a) The sequence $\{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$.

(b) $\{\frac{1+2n^2}{n^2}\} = \{3, \frac{9}{4}, \frac{19}{9}, \frac{33}{16}, \cdots\}$.

(c) $\{(-1)^n\} = \{-1, 1, -1, 1, \dots\}$.

(d) $\{\frac{1}{2^n}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$.

1.2 Graphs of Sequences

If we look at a sequence as a function, then we may consider its graph in the xy-plane. Since the domain of a sequence is the set of positive integers, the only points on the graph are

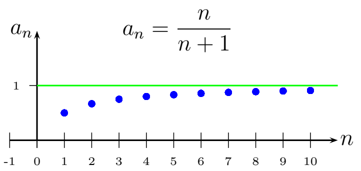
$$(1, x_1), (2, x_2), \cdots, (n, x_n), \cdots$$

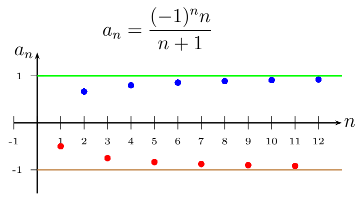
where x_n is the nth term of the sequence. We use the graph of a sequence to illustrate the behavior of the nth term as n increases without bound. The graphs of the following examples of sequences are shown below. Each of these sequences behaves differently as n gets larger.

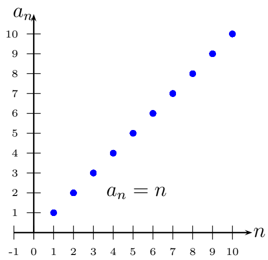
a) The sequence $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\right\}$ increase toward the number 1.

b) The sequence $\left\{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{4}, \dots, \frac{(-1)^n n}{n+1}, \dots\right\}$ oscillate between 1 and -1.

c) The sequence $\{1, 2, 3, \ldots, n, \ldots\}$ grows without a bound.



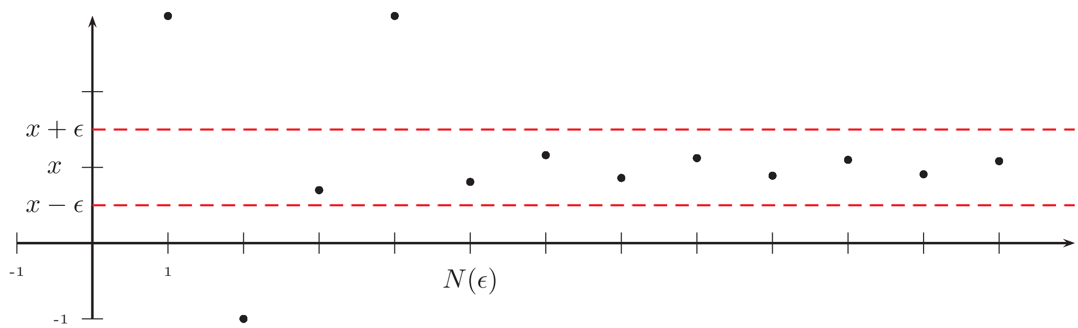




1.3 Limit of a Sequence

Definition 1.2: A sequence $\{x_n\}$ of real numbers is said to be ***converge*** to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a natural number $N = N(\epsilon) \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \epsilon$, and we write $\lim\limits_{n\to\infty} x_n = x$. If a sequence has a limit, we say that the sequence is ***convergent***; if it has no limit we say the sequence is ***divergent***.

Note 1.2: The definition says that all terms of the sequence beyond(after) the term $x_{N(\epsilon)}$ are within ϵ from x. The graph below demonstrates the definition. Notice that some of the terms that precede the $N(\epsilon)$ term may lie within ϵ of x. Every terms exceeds $N(\epsilon)$ must lie between $x - \epsilon$ and $x + \epsilon$.



Lemma 1.1: Let $\{x_n\}$ be sequence of real numbers such that $\lim\limits_{n\to\infty} x_n = x$, and $\lim\limits_{n\to\infty} x_n = y$. Then $x = y$.

Proof: Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = x$, then there exist $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Since $\lim\limits_{n\to\infty} x_n = y$, then there exist $N_2 \in \mathbb{N} \ni$ if $n > N_2 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Now, Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1, \Rightarrow |x_n - x| < \frac{\epsilon}{2}$ and if $n > N$, then $n > N_2, \Rightarrow |x_n - y| < \frac{\epsilon}{2}$. Then $|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $0 \leq |x - y| \leq \epsilon$. Thus $|x - y| = 0$. Therefore $x = y$.

Example 1.2: Let $a \in \mathbb{R}$ and for each $n \in \mathbb{N}$ let $x_n = a$ Prove that $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} a = a$.

Discussion: We start with $\epsilon > 0$ and want to find $N = N(\epsilon) \in \mathbb{N}$ such that if $n > n \Rightarrow |x_n - a| < \epsilon$. Now, $|x_n - a| = |a - a| = 0 < \epsilon$.

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$.

$$\text{if } n > N \Rightarrow |x_n - a| = |a - a| = 0 < \epsilon.$$

Thus $\lim\limits_{n\to\infty} a = a$.

Example 1.3: Let $a \in \mathbb{R}$ and $p \in \mathbb{R}$ with $p > 0$. Prove that $\lim\limits_{n\to\infty} \frac{a}{n^p} = 0$.

Discussion: We start with $\epsilon > 0$ and want to find $N = N(\epsilon) \in \mathbb{N}$ such that if $n > n \Rightarrow \left|\frac{a}{n^p} - 0\right| < \epsilon$. Now, $\left|\frac{a}{n^p} - 0\right| = \frac{|a|}{n^p}$. If $a = 0 = |a|$, then $\frac{|a|}{n^p} = 0 < \epsilon$ for all $n \in \mathbb{N}$ and hence we can choose N to be 1 for example. If $|a| \neq 0$, then

$$\begin{aligned} \frac{|a|}{n^p} < \epsilon \Leftrightarrow \frac{n^p}{|a|} > \frac{1}{\epsilon} & \quad \text{multiply both sides by } |a|. \\ \Leftrightarrow n^p > \frac{|a|}{\epsilon} & \quad \text{take the } p\text{-th root} \\ \Leftrightarrow n > \sqrt[p]{\frac{|a|}{\epsilon}} & \end{aligned}$$

Now, since $\sqrt[p]{\frac{|a|}{\epsilon}}$ may not by an natural number, we let $N = N(\epsilon) > \sqrt[p]{\frac{|a|}{\epsilon}}$.

Proof: Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $N > \sqrt[p]{\frac{|a|}{\epsilon}}$.

$$\begin{aligned} \text{if } n > N \Rightarrow n > N > \sqrt[p]{\frac{|a|}{\epsilon}} & \quad \text{take power } p \text{ for both sides} \\ \Rightarrow n^p > \frac{|a|}{\epsilon} & \quad \text{reverse the inequality} \\ \Rightarrow \frac{1}{n^p} < \frac{\epsilon}{|a|} & \quad \text{multiply both sides by } |a|. \\ \Rightarrow \frac{|a|}{n^p} < \epsilon & \\ \text{if } n > N \Rightarrow \left|\frac{a}{n^p} - 0\right| = \frac{|a|}{n^p} < \epsilon. & \end{aligned}$$

Thus $\lim_{n\to\infty} \frac{a}{n^p} = 0$.

Example 1.4: Prove that $\lim_{n\to\infty} \frac{2n^2+3}{3n^2-n} = \frac{2}{3}$.

Discussion: We start with $\epsilon > 0$ and want to find $N = N(\epsilon) \in \mathbb{N}$ such that if $n > n \Rightarrow \left|\frac{2n^2+3}{3n^2-n} - \frac{2}{3}\right| < \epsilon$.

$$\begin{aligned} \left|\frac{2n^2+3}{3n^2-n} - \frac{2}{3}\right| &= \left|\frac{6n^2+9-6n^2+2n}{9n^2-3n}\right| \\ &= \frac{2n+9}{9n^2-3n} && \text{Note that: } 2n+9 \le 2n+9n = 11n \\ &\le \frac{11n}{9n^2-3n} && \text{Note that: } 9n^2-3n \ge 9n^2-3n^2 \Leftrightarrow \frac{1}{9n^2-3n} \le \frac{1}{9n^2-3n^2} \\ &\le \frac{11n}{6n^2} = \frac{11}{6n}. \\ \text{Now, let } \frac{11}{6n} < \epsilon && & \Leftrightarrow n > \frac{11}{6\epsilon}. \end{aligned}$$

Now, since $\frac{11}{6\epsilon}$ may not by an natural number, we let $N = N(\epsilon) > \frac{11}{6\epsilon}$.

Proof: Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \frac{11}{6\epsilon}$.

$$\begin{aligned}
\text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < \frac{6\epsilon}{11} \\
&\Rightarrow \frac{11}{6n} < \epsilon \\
&\Rightarrow \left|\frac{2n^2+3}{3n^2-n} - \frac{2}{3}\right| < \frac{11}{6n} < \epsilon \\
\text{Now, if } n > N &\Rightarrow \left|\frac{2n^2+3}{3n^2-n} - \frac{2}{3}\right| < \epsilon. \\
\text{Therefore } &\lim_{n\to\infty} \frac{2n^2+3}{3n^2-n} = \frac{2}{3}.
\end{aligned}$$

Example 1.5: Prove that $\lim_{n\to\infty} \frac{n+3}{5n-1} = \frac{1}{5}$.

Discussion: We start with $\epsilon > 0$ and want to find $N = N(\epsilon) \in \mathbb{N}$ such that if $n > n \Rightarrow \left|\frac{n+3}{5n-1} - \frac{1}{5}\right| < \epsilon$.

$$\begin{aligned}
\left|\frac{n+3}{5n-1} - \frac{1}{5}\right| &= \left|\frac{5n+15-5n+1}{25n-5}\right| \\
&= \frac{16}{25n-5} \qquad \text{Note that: } -5 \geq -5n \Leftrightarrow 25n-5 \geq 25n-5n \Leftrightarrow \frac{1}{25n-5} \leq \frac{1}{20n} \\
&\leq \frac{16}{20n} \\
&= \frac{4}{5n}. \\
\text{Now, let } \frac{4}{5n} &< \epsilon \\
\Leftrightarrow n &> \frac{4}{5\epsilon}.
\end{aligned}$$

Now, since $\frac{4}{5\epsilon}$ may not by an natural number, we let $N = N(\epsilon) > \frac{11}{6\epsilon}$.

Proof: Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \frac{4}{5\epsilon}$.

$$\begin{aligned}
\text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < \frac{5\epsilon}{4} \\
&\Rightarrow \frac{4}{5n} < \epsilon \\
&\Rightarrow \left|\frac{n+3}{5n-1} - \frac{1}{5}\right| < \frac{4}{5n} < \epsilon \\
\text{Now, if } n > N &\Rightarrow \left|\frac{n+3}{5n-1} - \frac{1}{5}\right| < \epsilon. \\
\text{Therefore } &\lim_{n\to\infty} \frac{n+3}{5n-1} = \frac{1}{5}.
\end{aligned}$$

Theorem 1.1: *[]*

1. If $a \in \mathbb{R}$ and $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.
2. If $a \in \mathbb{R}$ and $a > 0$, then $\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} a^{\frac{1}{n}} = 0$.

3. $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{n} = 1.$

Proof:

1. *Discussion:* If $a = 0$, then $\lim_{n\to\infty} a^n = \lim_{n\to\infty} 0 = 0$. Assume that $a \neq 0$. Since $|a| < 1$, then $|a| = \dfrac{1}{1+b}$ where $b > 0$. By binomial theorem $(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k = 1 + nb + \cdots + b^n \geq 1 + nb > nb$. Hence $|a|^n = \left(\dfrac{1}{1+b}\right)^n = \dfrac{1}{(1+b)^n} < \dfrac{1}{nb}$.

$$\begin{aligned} |a^n - 0| &= |a|^n \\ &< \frac{1}{nb} \\ \text{Now, let } \frac{1}{bn} &< \epsilon \\ \Leftrightarrow n &> \frac{1}{b\epsilon}. \end{aligned}$$

Let $N = N(\epsilon) > \dfrac{1}{b\epsilon}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \dfrac{1}{b\epsilon}$.

$$\begin{aligned} \text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < b\epsilon \\ &\Rightarrow \frac{1}{bn} < \epsilon \\ &\Rightarrow |a^n - 0| < \frac{1}{bn} < \epsilon \\ \text{Now, if } n > N &\Rightarrow |a^n - 0| < \epsilon. \\ \text{Therefore } &\lim_{n\to\infty} a^n = 0. \end{aligned}$$

2. Case I: $a > 1$ *Discussion:* If $a > 1$, then $\sqrt[n]{a} > 1$ and hence $\sqrt[n]{a} = 1 + b_n$ for some $b_n > 0$. Hence $a = (1+b_n)^n \geq 1 + nb_n$. Thus $a - 1 \geq nb_n$ and hence $b_n \leq \dfrac{a-1}{n}$. Now, $0 < \sqrt[n]{a} - 1 = b_n$.

$$\begin{aligned} \left|\sqrt[n]{a} - 1\right| &= b_n \\ &< \frac{a-1}{n} \\ \text{Now, let } \frac{a-1}{n} &< \epsilon \\ \Leftrightarrow n &> \frac{a-1}{\epsilon}. \end{aligned}$$

Let $N = N(\epsilon) > \dfrac{a-1}{\epsilon}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \dfrac{a-1}{\epsilon}$.

$$\begin{aligned} \text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < \frac{\epsilon}{a-1} \\ &\Rightarrow \frac{1-a}{n} < \epsilon \\ &\Rightarrow \left|\sqrt[n]{a} - 1\right| < \frac{a-1}{n} < \epsilon \\ \text{Now, if } n > N &\Rightarrow \left|\sqrt[n]{a} - 1\right| < \epsilon. \\ \text{Therefore } &\lim_{n\to\infty} \sqrt[n]{a} = 1. \end{aligned}$$

Case II: $a < 1$ *Discussion:* If $a < 1$, then $\sqrt[n]{a} < 1$ and hence $\sqrt[n]{a} = \dfrac{1}{1+b_n}$ for some $b_n > 0$. Hence $a = \dfrac{1}{(1+b_n)^n} \leq \dfrac{1}{1+nb_n} \leq \dfrac{1}{nb_n}$ and hence $0 < b_n \leq \dfrac{1}{na}$. Also, we have $1+b_n > 1$ and hence $\dfrac{1}{1+b_n} < 1$. Thus if we multiply the last inequality by $b_n > 0$, we get $\dfrac{b_n}{1+b_n} < b_n$. Now, $0 < 1 - \sqrt[n]{a} = 1 - \dfrac{1}{1+b_n} = \dfrac{b_n}{1+b_n} < b_n << \dfrac{1}{na}$.

$$\begin{aligned} \left|1 - \sqrt[n]{a}\right| &< b_n \\ &< \frac{1}{an} \\ \text{Now, let } \frac{1}{an} &< \epsilon \\ \Leftrightarrow n &> \frac{1}{a\epsilon}. \end{aligned}$$

Let $N = N(\epsilon) > \dfrac{1}{a\epsilon}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \dfrac{1}{a\epsilon}$.

$$\begin{aligned} \text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < a\epsilon \\ &\Rightarrow \frac{a}{n} < \epsilon \\ &\Rightarrow \left|\sqrt[n]{a} - 1\right| < \frac{a}{n} < \epsilon \\ \text{Now, if } n > N &\Rightarrow \left|\sqrt[n]{a} - 1\right| < \epsilon. \\ \text{Therefore } &\lim_{n\to\infty} \sqrt[n]{a} = 1. \end{aligned}$$

Case II: $a = 1$ If $a = 1$, then $\sqrt[n]{a} = 1$ and hence $\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} 1 = 1$.

3. *Discussion:* If $n > 1$, then $\sqrt[n]{n} > 1$ and hence $\sqrt[n]{n} = 1 + b_n$ for some $b_n > 0$. Hence $n = (1+b_n)^n = 1 + nb_n + \dfrac{1}{2}n(n-1)b_n^2 + \ldots \geq 1 + \dfrac{1}{2}n(n-1)b_n^2$. Thus $n - 1 \geq \dfrac{n(n-1)}{2}b_n^2$ and hence $b_n^2 \leq \dfrac{2}{n}$. Now, $0 < \sqrt[n]{n} - 1 = b_n$.

$$\begin{aligned} \left|\sqrt[n]{n} - 1\right| &= b_n \\ &< \sqrt{\frac{2}{n}} \\ \text{Now, let } \sqrt{\frac{2}{n}} &< \epsilon \Leftrightarrow n > \frac{2}{\epsilon^2}. \end{aligned}$$

Let $N = N(\epsilon) > \dfrac{2}{\epsilon^2}$. Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \dfrac{2}{\epsilon^2}$.

$$\begin{aligned} \text{Now, if } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < \frac{\epsilon^2}{2} \\ &\Rightarrow \frac{2}{n} < \epsilon^2 \\ &\Rightarrow \sqrt{\frac{2}{n}} < \epsilon \\ &\Rightarrow \left|\sqrt[n]{n} - 1\right| < \sqrt{\frac{2}{n}} < \epsilon \\ \text{Now, if } n > N &\Rightarrow \left|\sqrt[n]{n} - 1\right| < \epsilon. \qquad \text{Therefore } \lim_{n\to\infty} \sqrt[n]{n} = 1. \end{aligned}$$

Example 1.6: Let $a, b \in \mathbb{R}$ with $a \neq b$ and let $x_n = \begin{cases} a, & \text{if } n \text{ is odd;} \\ b, & \text{if } n \text{ is even.} \end{cases}$ Prove that $\{x_n\}$ is divergent.

Proof: Suppose that $\{x_n\}$ is convergent, then there exist $l \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = l$. Then for any $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - l| < \epsilon$. Now, since $a \neq b$, then $|a - b| > 0$ and hence if $\epsilon_0 = \dfrac{|a-b|}{4} > 0$ then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - l| < \dfrac{|a-b|}{4}$ If $n > N$ and n is even the $|b - l| = |x_n - l| < \dfrac{|a-b|}{4}$ also, if $n > N$ and n is odd the $|a - l| = |x_n - l| < \dfrac{|a-b|}{4}$ Now, if $n > N$ $|a - b| = |a - l - b + l| = |(a - l) - (b - l)| \leq |a - l| + |b - l| < \dfrac{|a-b|}{4} + \dfrac{|a-b|}{4} = \dfrac{|a-b|}{2}$. Hence $|a - b| < \dfrac{|a-b|}{2}$ contradiction. ■

