Sequences Limit Theorems

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November 3, 2012

2.1 Bounded Sequence

Definition 2.1: A sequence $\{x_n\}_{n=1}^{\infty}$, of a real numbers is said to be ***bounded*** if there exists a real number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Example 2.1:

(a) $\left\{\frac{1}{n}\right\}$ is bounded since $\left|\frac{1}{n}\right| \leq 1$.

(b) $\left\{\frac{1+2n^2}{n^2}\right\}$ is bounded since $\left|\frac{1+2n^2}{n^2}\right| \leq 3$.

(c) $\{(-1)^n\}$ is bounded since $|(-1)^n| \leq 1$.

(d) $\{2^n\}$ is unbounded since for any real number $M > 0 \; \exists \; n \in \mathbb{N} \ni n > M$ and $2^n > n > M$.

Theorem 2.1: ***[Convergent Sequence is Bounded]***

A converge sequence of real numbers is bounded.

Proof: Let $\{x_n\}$ be sequence of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. Since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that if $n > N, \Rightarrow |x_n - x| < 3$. Thus, if $n > N, \Rightarrow |x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 3 + |x|$. Let $M = \max\{|x_1|, |x_2|, ...|x_N|, 3 + |x|\} > 0$. Now, if $n > N, \Rightarrow |x_n| < 3 + |x| \leq M$, and if $n \leq N, \Rightarrow |x_n| \leq M$. Thus $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Note 2.1: Remember that the negation of theorem is also true. Hence unbounded sequence is divergent. Also note the the converse of this theorem is false. There is a divergent bounded sequence. for example $\{(-1)^n\}$.

2.2 Arithmetic Operations On Sequences

Theorem 2.2: ***[Addition, difference, and Multiplication]***

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ and $\lim_{n\to\infty} y_n = y \in \mathbb{R}$. Let $c \in \mathbb{R}$. Then

(a) $\lim_{n\to\infty} (cx_n) = cx$.

(b) $\lim_{n\to\infty} (x_n + y_n) = x + y$.

(c) $\lim_{n\to\infty} (x_n - y_n) = x - y$.

(d) $\lim_{n\to\infty} (x_n y_n) = xy$.

Proof:

(a) If $c = 0$, there is nothing to prove. Assume $c \neq 0$. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{|c|}$. Thus if $n > N, \Rightarrow |cx_n - cx| = |c(x_n - x)| = |c||x_n - x| < |c|\frac{\epsilon}{|c|} = \epsilon$. Therefore $\lim_{n\to\infty}(cx_n) = cx$.

(b) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Also, since $\lim_{n\to\infty} y_n = y$, therefore there exists $N_2 \in \mathbb{N} \ni$ if $n > N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$. Now, if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{2}$, and $|y_n - y| < \frac{\epsilon}{2}$.

$$\begin{aligned}\text{Thus if } n > N, \Rightarrow |(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty}(x_n + y_n) = x + y$.

(c) Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, then there exist $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Also, since $\lim_{n\to\infty} y_n = y$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$. Let $N = max\{N_1, N_2\} \in \mathbb{N}$. Now, if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{2}$, and $|y_n - y| < \frac{\epsilon}{2}$.

$$\begin{aligned}\text{Thus if } n > N, \Rightarrow |(x_n - y_n) - (x - y)| = |(x_n - x) - (y_n - y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty}(x_n - y_n) = x - y$.

(d) Let $\epsilon > 0$ be given. Now, since $\{x_n\}$ converges, then it is bounded. Then there exists $M \in \mathbb{R}^+$ such that $|x_n| \leq M$, for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = x$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2(|y| + 1)}$. Also, since $\lim_{n\to\infty} y_n = y$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2M}$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$. Now, if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{2(|y| + 1)}$, and $|y_n - y| < \frac{\epsilon}{2M}$.

$$\begin{aligned}\text{Thus if } n > N \Rightarrow |(x_n y_n) - (xy)| = |x_n y_n - x_n y + x_n y - xy| &\leq |x_n(y_n - y)| + |y(x_n - x)| \\ &= |x_n||y_n - y| + |y||x_n - x| \\ &< M|y_n - y| + |y||x_n - x| \\ &< M\frac{\epsilon}{2M} + |y|\frac{\epsilon}{2(|y| + 1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty}(x_n y_n) = xy$. ■

Theorem 2.3: []

Let $\{x_n\}$ be sequence of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, and $x \neq 0$. Then

(a) If $1 < \beta$ there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n| \geq \frac{|x|}{\beta}$.

(b) If $x_n \neq 0$, for all $n \in \mathbb{N}$, then $\inf\{|x_n| : n \in \mathbb{N}\} > 0$.

(c) If $x_n \neq 0$, for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

Proof:

(a) Let $\epsilon = \frac{(\beta-1)|x|}{\beta} > 0$. Since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \frac{(\beta-1)|x|}{\beta}$. Now,if $n > N \Rightarrow |x| - |x_n| \leq |x_n - x| < \frac{(\beta-1)|x|}{\beta}$. Hence, if $n > N \Rightarrow |x| - |x_n| < \frac{(\beta-1)|x|}{\beta}$. Thus, if $n > N \Rightarrow |x| - \frac{(\beta-1)|x|}{\beta} < |x_n|$. Therefore, if $n > N \Rightarrow \frac{|x|}{\beta} < |x_n|$.

(b) Let $\epsilon > 0$ be given.By **part (a)**,for $\beta = 2$, then there exists $N \in \mathbb{N} \ni$ if $n > N \Rightarrow |x_n| > \frac{|x|}{2}$. Now, since $x_n \neq 0$, for all $n \in \mathbb{N}$, then $|x_n| > 0$, for all $n \in \mathbb{N}$. Hence $m = \min\{|x_1|, |x_2|, \cdots, |x_N|, \frac{|x|}{2}\} > 0$. Thus $|x_n| \geq m$ for all $n \in \mathbb{N}$. Hence $\inf\{|x_n| : n \in \mathbb{N}\} \geq m > 0$.

(c) Let $\epsilon > 0$ be given. Since $x_n \neq 0$, for all $n \in \mathbb{N}$, then by **part (b)** $m = \inf\{|x_n| : n \in \mathbb{N}\} > 0$ and hence $|x_n| \geq m$. Thus $\frac{1}{|x_n|} \leq \frac{1}{m}$. Also, Since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < m|x|\epsilon$.

$$\text{Thus if } n > N \Rightarrow \left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x x_n}\right| = \frac{|x_n - x|}{|x||x_n|} < \frac{|x_n - x|}{m|x|} < \frac{m|x|\epsilon}{m|x|} = \epsilon.$$

Therefore $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

Corollary 2.1: Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ and $\lim_{n\to\infty} y_n = y \in \mathbb{R}$. Let $c \in \mathbb{R}$. If $y_n \neq 0$ for all $n \in \mathbb{N}$, and $y \neq 0$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof: Since $y_n \neq 0$ for all $n \in \mathbb{N}$, and $y \neq 0$, then $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$. Now,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \left[x_n \frac{1}{y_n}\right] = \lim_{n\to\infty} x_n \lim_{n\to\infty} \frac{1}{y_n} = x\frac{1}{y} = \frac{x}{y}.$$

Theorem 2.4: *[Squeeze Theorem]*

Suppose that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are sequences of real numbers such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \in \mathbb{R}$. Then $\lim_{n\to\infty} y_n = L$.

Proof: Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = L$, then there exists $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - L| < \epsilon$. Also, since $\lim_{n\to\infty} z_n = L$, then there exists $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow |z_n - L| < \epsilon$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$. Now, if $n > N \Rightarrow |x_n - L| < \epsilon$, and $|z_n - L| < \epsilon$.

If $n > N \Rightarrow |x_n - L| < \epsilon$.

If $n > N \Rightarrow -\epsilon < x_n - L < \epsilon$.

If $n > N \Rightarrow L - \epsilon < x_n < L + \epsilon$.

Also, if $n > N \Rightarrow |z_n - L| < \epsilon$.

If $n > N \Rightarrow -\epsilon < z_n - L < \epsilon$.

If $n > N \Rightarrow L - \epsilon < z_n < L + \epsilon$.

Hence, if $n > N \Rightarrow L - \epsilon < x_n \le y_n \le z_n < L + \epsilon$.

Thus, if $n > N \Rightarrow -\epsilon < y_n - L < \epsilon$.

Therefore, if $n > N \Rightarrow |y_n - L| < \varepsilon$. Thus $\lim_{n\to\infty} y_n = L$.

Example 2.2: Prove the following

(a) $\lim_{n\to\infty} (\frac{\sin n}{n}) = 0$.

(b) $\lim_{n\to\infty} (\frac{n!}{n^n}) = 0$.

Solution:

(a) Since $-1 \le \sin n \le 1$, then $\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$. Since $\lim_{n\to\infty} (\frac{-1}{n}) = 0 = \lim_{n\to\infty} (\frac{1}{n})$, then by the Squeeze Theorem , $\lim_{n\to\infty} (\frac{\sin n}{n}) = 0$.

(b) Since

$$0 < \frac{n!}{n^n} = \frac{n.(n-1)\cdots .2.1}{\underbrace{n.n\ \cdots .n}_{n}} = \frac{n}{n}\frac{n-1}{n}\cdots\frac{2}{n}\frac{1}{n} \le 1.1\cdots\frac{1}{n} = \frac{1}{n},$$

and $\lim_{n\to\infty}(0) = 0 = \lim_{n\to\infty}(\frac{1}{n})$, then by the Squeeze Theorem , $\lim_{n\to\infty}(\frac{n!}{n^n}) = 0$.

Lemma 2.1: Let $\{x_n\}$ be sequence of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, then $\lim_{n\to\infty} |x_n| = |x|$.

Proof: We have shown in class that $||a| - |b|| \le |a - b|,\ \forall a, b \in \mathbb{R}$.

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, therefore there exists $N \in \mathbb{N} \ni |x_n - x| < \epsilon$. Now,if $n > N \Rightarrow ||x_n| - |x|| \le |x_n - x| < \epsilon$. Thus $\lim_{n\to\infty} |x_n| = |x|$.

Note 2.2: The converse of the lemma is not true there is a divergent sequence such that the sequence of the absolute value is convergent. For example, let $x_n = (-1)^n$, then $x_n = (-1)^n$ is divergent. Now, $|x_n| = |(-1)^n| = 1$ and hence $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 1 = 1$.

Lemma 2.2: Let $\{x_n\}$ be sequence of real numbers such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, and $x_n \ge 0$. Then

(a) $x \ge 0$, and

(b) $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof:

(a) Suppose that $x < 0$, then let $\epsilon = -x > 0$, since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \epsilon$.

If $n > N \Rightarrow -\varepsilon < x_n - x < \epsilon$.

If $n > N \Rightarrow x - (-x) < x_n < x + (-x)$.

If $n > N \Rightarrow 2x < x_n < 0$, contradiction. Thus $x \geq 0$.

(b) Using the fact $|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a-b|}$, $\forall a, b \in \mathbb{R}^+$.

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, then there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon^2$. Now, if $n > N \Rightarrow |\sqrt{x_n} - \sqrt{x}| \leq \sqrt{|x_n - x|} < \sqrt{\epsilon^2} = \epsilon$. Thus $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.