Chapter Six

Discrete Probability

6-1Introduction

- Many decisions in business, insurance, and other real-life situations are made by assigning probabilities to all possible outcomes pertaining to the situation and then evaluating the results.

- This chapter explains the concepts and applications of probability distributions. In addition, a special probability distribution, **binomial distribution**, is explained.

6-2 Discrete Probability Distribution

- A random variable is a variable whose values are determined by chance.

- A **discrete probability distribution** consists of the values a random variable can assume and the corresponding probabilities of the values. The probabilities are determined theoretically or by observation.

6-2 Discrete Probability Distribution

- Example:
  - Construct a probability distribution for rolling a single die.

    Since the sample space is ***S={1,2,3,4,5,6}*** and each outcome has a probability ***1/6*** , the distribution will be

Outcome x	1	2	3	4	5	6
Probability $P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

6-2 Discrete Probability Distribution

- Example:
  - Construct a probability distribution for the sample space for tossing three coins.

Number of heads x	0	1	2	3
Probability $P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

6-2 Discrete Probability Distribution

- Example:
  - During the summer months, a rental agency keeps track of the number of chain saws it rents each day during a period of 90 days. The number of saws rented per day is represented by the variable *X*. The results are shown here. Construct a probability distribution.

x	0	1	2	Total
# of days	45	30	15	90

x	0	1	2
$P(x)$	$\frac{45}{90} = 0.5$	$\frac{30}{90} = 0.333$	$\frac{15}{90} = 0.167$

6-3 Requirements for a Probability Distribution

- The sum of the probabilities of all the events in the sample space must equal 1;

$$\sum \boldsymbol{P(x)} = \mathbf{1}$$

- The probability of each event in the sample space must be between or equal to 0 and 1;

$$\mathbf{0} \leq \boldsymbol{P(x)} \leq \mathbf{1}$$

6-3 Requirements for a Probability Distribution

- <u>Example</u>:
  - Determine whether each distribution is a probability distribution.
    - a-

x	0	5	10	15	20
$P(x)$	1/5	1/5	1/5	1/5	1/5

Yes, it is a probability distribution.

- b-

x	0	2	4	6
$P(x)$	-1.0	1.5	0.3	0.2

No, it is not a probability distribution, since ***P(X)*** cannot be ***1.5*** or ***-1.0***

6-3 Requirements for a Probability Distribution

- c-

x	1	2	3	4
$P(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{9}{16}$

Yes, it is a probability distribution.

- d-

x	2	3	7
$P(x)$	0.5	0.3	0.4

No, it is not, since ***P(x)=1.2***

6-4 Mean of a Probability Distribution

- In order to find the mean for a probability distribution, one must multiply each possible outcome by its corresponding probability and find the sum of the products.

$$\mu = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n) = \sum xp(x)$$

- Example:
- In a family with two children, find the mean of the number of children who will be girls.

The probability distribution is

# of girls x	0	1	2	$\sum$
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1
$xP(x)$	0	$\frac{1}{2}$	$\frac{2}{4}$	1

6-4 Mean of a Probability Distribution

- Example:
  - If three coins are tossed, find the mean of the number of heads that occur.

  The probability distribution is

# of heads x	0	1	2	3	$\sum$
$P(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1
$xP(x)$	0	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{3}{8}$	$\frac{12}{8}=\frac{3}{2}$

6-4 Mean of a Probability Distribution

- <u>Example</u>:
  - The probability distribution shown represents the number of trips of five nights or more that American adults take per year.

# of trips x	0	1	2	3	4
$P(x)$	0.06	0.70	0.20	0.03	0.01

Find the mean.

# of trips x	0	1	2	3	4	$\sum$
$P(x)$	0.06	0.70	0.20	0.03	0.01	1
$xP(x)$	0	0.70	0.40	0.09	0.04	1.23

6-5 Variance of a Probability Distribution

- The variance of a probability distribution is found by multiplying the square of each outcome by its corresponding probability, summing those products, and subtracting the square of the mean.

  - The formula for calculating the variance is:

$$\boldsymbol{\sigma^2 = \sum x^2 p(x) - \mu^2}$$

  - The formula for the standard deviation is:

$$\boldsymbol{\sigma = \sqrt{\sigma^2}}$$

6-5 Variance of a Probability Distribution

- Example:
  - The probability distribution for the number of spots that appear when a die is tossed

Outcome x	1	2	3	4	5	6
Probability $\boldsymbol{P}(\boldsymbol{x})$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Find the variance and standard deviation of the number of spots.

6-5 Variance of a Probability Distribution

Outcome x	1	2	3	4	5	6	$\sum$
Probability $\boldsymbol{P(x)}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1
$xP(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{21}{6}$
$\boldsymbol{x^2P(x)}$	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{9}{6}$	$\frac{16}{6}$	$\frac{25}{6}$	$\frac{36}{6}$	$\frac{91}{6}$

$$\boldsymbol{\mu = \sum xP(x) = \frac{21}{6}}$$

$$\boldsymbol{\sigma^2 = \sum x^2P(x) - \mu^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{441}{36} = \frac{546 - 441}{36}}$$

$$\boldsymbol{= \frac{105}{36} = 2.92}$$

$$\boldsymbol{\sigma = \sqrt{\sigma^2} = \sqrt{2.92} = 1.71}$$

6-5 Variance of a Probability Distribution

- <u>Example</u>:
  - Five balls numbered 0, 2, 4, 6 and 8 are placed in a bag. After the balls are mixed, one is selected, its number is noted and then it is replaced. If this experiment is repeated many times, and the probability distribution is

# on ball x	0	2	4	6	8
Probability $\boldsymbol{P}(\boldsymbol{x})$	k	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

find the missing value (k) , mean, variance and standard deviation of the numbers on the balls.

6-5 Variance of a Probability Distribution

$$\boldsymbol{k+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=1 \rightarrow k+\frac{4}{5}=1 \rightarrow k=1-\frac{4}{5} \rightarrow k=\frac{1}{5}}$$

# on ball x	0	2	4	6	8	$\sum$
Probability $\boldsymbol{P(x)}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
$xP(x)$	0	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{6}{5}$	$\frac{8}{5}$	$\frac{20}{5}=4$
$\boldsymbol{x^2P(x)}$	0	$\frac{4}{5}$	$\frac{16}{5}$	$\frac{36}{5}$	$\frac{64}{5}$	$\frac{120}{5}=24$

$$\boldsymbol{\mu=\sum xP(x)=4}$$

$$\boldsymbol{\sigma^2=\sum x^2P(x)-\mu^2=24-(4)^2=24-16=8}$$

$$\boldsymbol{\sigma=\sqrt{\sigma^2}=\sqrt{8}=2.83}$$

6-6 The Binomial Distribution

- Many types of probability problems have only two possible outcomes or they can be reduced to two outcomes.

- Examples:
  - When a coin is tossed it can land on heads or tails.
  - When a baby is born it is either a boy or girl.
  - A multiple-choice question can be classified as correct or incorrect.

6-7 The Binomial Experiment

- The binomial experiment is a probability experiment that satisfies these requirements:
  - Each trial can have only two possible outcomes - success or failure.
  - There must be a fixed number of trials.
  - The outcomes of each trial must be independent of each other.
  - The probability of a success must remain the same for each trial.

6-7 The Binomial Experiment

- The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a <u>**binomial distribution**</u> which is the probability of exactly X successes in n trials

$$P(x) = \frac{n!}{x!\,(n-x)!}p^x q^{n-x}$$

where

- ***p*** The numerical probability of success
- ***q*** The numerical probability of failure

$$p + q = 1$$

- ***n*** The number of trials
- ***x*** The number of successes ***x=0, 1, 2, …, n***

6-7 The Binomial Experiment

- Example:
  - A coin is tossed 3 times. Find the probability of getting exactly two heads.

This can solved using the sample space

HHH,HHT,HTH,THH,HTT,THT,TTH,TTT

There are three ways of getting 2 heads.

$$\boldsymbol{P(getting\ 2\ heads) = \frac{n(getting\ 2\ heads)}{n(S)} = \frac{3}{8} = 0.375}$$

6-7 The Binomial Experiment

- Or using the binomial distribution as following
  - we have fixed number of trials (three), so ***n=3***
  - there are two outcomes for each trial, ***H*** or ***T***
  - the outcomes are independent of one another
  - the probability of success (heads) is $\frac{1}{2}$, so $p = \frac{1}{2} \Longrightarrow q = 1 - \frac{1}{2} = \frac{1}{2}$

here ***x=2*** since we need to find the probability of getting 2 heads,

$$P(x = 2) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x} = \frac{3!}{(2!)(1!)} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1$$

$$= \frac{(3)(2)(1)}{(2)(1)(1)} \left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{6}{16} = \frac{3}{8} = 0.375$$

6-7 The Binomial Experiment

- Example:
  - A survey found that one out of five Americans says he or she has visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 would have visited a doctor last month.

In this case $\boldsymbol{n = 10, X = 3, p = \frac{1}{5} \Rightarrow q = 1 - \frac{1}{5} = \frac{4}{5}}$

$$\boldsymbol{P(x = 3) = \frac{n!}{x!\,(n-x)!} p^x q^{n-x} = \frac{10!}{(3!)(7!)} \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7}$$

$$\boldsymbol{= \frac{(10)(9)(8)(7!)}{(3)(2)(1)(7!)} \left(\frac{1}{125}\right)\left(\frac{16384}{78125}\right) = \frac{11796480}{58593750} = 0.201}$$

6-7 The Binomial Experiment

- Example:
  - A survey from Teenage Research Unlimited (Northbrook, Illinois) found that 30% of teenage consumers receive their spending money from part-time jobs. If 5 teenagers are selected at random, find the probability that at least 3 of them will have part-time jobs.

In this case $\boldsymbol{n = 5, p = 0.3 \Longrightarrow q = 1 - 0.3 = 0.7}$

$$\boldsymbol{P(at\ least\ 3\ have\ part - time\ job) = P(x \geq 3)}$$

$$\boldsymbol{= P(x = 3) + P(x = 4) + P(x = 5)}$$

$$\boldsymbol{= \frac{5!}{(3!)(2!)}(0.3)^3(0.7)^2 + \frac{5!}{(4!)(1!)}(0.3)^4(0.7)^1 + \frac{5!}{(5!)(0!)}(0.3)^5(0.7)^0}$$

$$\boldsymbol{= 0.132 + 0.028 + 0.002}$$

$$\boldsymbol{= 0.162}$$

6-8 Binomial Distribution Properties

- The mean, variance, and standard deviation of a variable that has the binomial distribution can be found by using the following formulas.
  - Mean

$$\boldsymbol{\mu = np}$$

  - Variance

$$\boldsymbol{\sigma^2 = npq}$$

  - Standard deviation

$$\boldsymbol{\sigma = \sqrt{\sigma^2}}$$

6-8 Binomial Distribution Properties

- Example:
  - A coin is tossed 4 times. Find the mean, variance and standard deviation of the number of heads that will be obtained.

In this case $\boldsymbol{n = 4, p = \frac{1}{2} \Longrightarrow q = 1 - \frac{1}{2} = \frac{1}{2}}$

$$\boldsymbol{\mu = np = (4)\left(\frac{1}{2}\right) = 2}$$

$$\boldsymbol{\sigma^2 = npq = (4)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1}$$

$$\boldsymbol{\sigma = \sqrt{\sigma^2} = \sqrt{1} = 1}$$

6-8 Binomial Distribution Properties

- Example:
  - A die is rolled 480 times. Find the mean, variance and standard deviation of the number of 2s that will be obtained.

In this case $\boldsymbol{n = 480, p = \frac{1}{6} \Longrightarrow q = 1 - \frac{1}{6} = \frac{5}{6}}$

$$\boldsymbol{\mu = np = (480)\left(\frac{1}{6}\right) = 80}$$

$$\boldsymbol{\sigma^2 = npq = (480)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = 66.7}$$

$$\boldsymbol{\sigma = \sqrt{\sigma^2} = \sqrt{66.7} = 8.2}$$