Riemann Sums

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Definition 3.1: Let Let $f : [a, b] \to \mathbb{R}$ be bounded function , and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. For each interval $[x_{k-1}, x_k]$ choose $t_k \in [x_{k-1}, x_k]$. Then the number $S(f, P) = \sum\limits_{k=1}^{n} f(t_k)\triangle x_k$ is called a *Riemann sum* of f over P determined by $\{t_k\}$.

Example 3.1: Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P_n = \left\{x_k \mid x_k = a + \dfrac{k(b-a)}{n}, k = 0, 1, \ldots, n\right\}$, $\forall\ n \in \mathbb{N}$ be a partition of $[a, b]$. Let $t_k = x_{k-1}, k = 1, ..., n$ then Riemann Sum is $S(f, P) = \sum\limits_{k=1}^{n} f(x_{k-1})\triangle x_k$. Also if we choose $t_k = \frac{x_k + x_{k-1}}{2}$, then Riemann Sum is $S(f, P) = \sum\limits_{k=1}^{n} f\left(\frac{x_k+x_{k-1}}{2}\right)\triangle x_k$. The two sums may not be equal.

Theorem 3.1: []

Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$. Then f is integrable on $[a, b]$ if and only if there is a real number $A \in \mathbb{R}$ such that for $\varepsilon > 0$ there is a partition P_ε such that if P is any partition of $[a, b]$ with $P_\varepsilon \subseteq P$ and if $S(f, P)$ is any Riemann Sum for f over P then $|S(f, P) - A| < \varepsilon$. If the condition is satisfied, then $A = \int\limits_a^b f$.

Proof: ($\Rightarrow$) Suppose f is integrable on $[a, b]$. Let $A = \int\limits_a^b f$ and $\epsilon > 0$ be given. There is a partition P_ε of $[a, b]$ such that $U(f, P_\varepsilon) - L(f, P_\varepsilon) < \epsilon$. Let P any partition of $[a, b]$ such that $P_\varepsilon \subseteq P$. Then $L(f, P_\varepsilon) \le L(f, P) \le S(f, P) \le U(f, P) \le U(f, P_\varepsilon) < L(f, P_\varepsilon) + \epsilon$.
Also $L(f, P_\varepsilon) \le L(f, P) \le \int\limits_a^b f \le U(f, P) \le U(f, P_\varepsilon) < L(f, P_\varepsilon) + \epsilon$. Hence $|S(f, P) - \int\limits_a^b f| < \varepsilon$.

($\Leftarrow$) Let $\epsilon > 0$ be given. By the condition there is a partition$P_\varepsilon = \{a = x_0, x_1, ..., x_n = b\}$ such that $|S(f, P_\varepsilon) - A| < \frac{\varepsilon}{4}$. For each $k = 1, 2, ..., n$ choose $t_k, u_k \in [x_{k-1}, x_k]$ such that $M_k(f) - \frac{\epsilon}{4(b-a)} < f(t_k)$ and $f(u_k) < m_k(f) + \frac{\epsilon}{4(b-a)}$. Thus $M_k(f) - m_k(f) < f(t_k) + \frac{\epsilon}{4(b-a)} - f(u_k) + \frac{\epsilon}{4(b-a)} = f(t_k) - f(u_k) + \frac{\epsilon}{2(b-a)}$. Now,

$$\begin{aligned}
U(f, P_\varepsilon) - L(f, P_\varepsilon) &= \sum_{k=1}^{n} (M_k(f) - m_k(f))\triangle x_k \\
&< \sum_{k=1}^{n} \left[f(t_k) - f(u_k) + \frac{\epsilon}{2(b-a)}\right] \triangle x_k \\
&= \sum_{k=1}^{n} f(t_k)\triangle x_k - \sum_{k=1}^{n} f(u_k)\triangle x_k + \sum_{k=1}^{n} \frac{\epsilon}{2(b-a)}\triangle x_k \\
&= \left[\sum_{k=1}^{n} f(t_k)\triangle x_k - A\right] - \left[\sum_{k=1}^{n} f(u_k)\triangle x_k - A\right] + \sum_{k=1}^{n} \frac{\epsilon}{2(b-a)}\triangle x_k \\
&= [S(f, P_\varepsilon) - A] - [S(f, P_\varepsilon) - A] + \frac{\epsilon}{2(b-a)}(b-a) \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus f is integrable on $[a,b]$.

To show that $A=\int_a^b f$:

Let $\epsilon>0$ be given. By the condition there is a partition P_1 such that if Q is any partition of $[a,b]$ with $P_1\subseteq Q$ then $|S(f,Q)-A|<\frac{\epsilon}{3}$. Also since f is integrable there is a partition P_2 of $[a,b]$ such that $U(f,P_2)-L(f,P_2)<\frac{\epsilon}{3}$. Let $P=P_1\cup P_2$ then $U(f,P)-L(f,P)<\frac{\epsilon}{3}$ and $|S(f,P)-A|<\frac{\epsilon}{3}$. Thus $U(f,P)<L(f,P)+\frac{\epsilon}{3}$. Note that $L(f,P)\le S(f,P)\le U(f,P)$. Hence $L(f,P)\le S(f,P)\le U(f,P)<L(f,P)+\frac{\epsilon}{3}$. Therefore $|S(f,P)-L(f,P)|<\frac{\epsilon}{3}$. Also we have $L(f,P)\le\int_a^b f\le U(f,P)$. Hence $L(f,P)\le\int_a^b f\le U(f,P)<\frac{\epsilon}{3}$. Therefore $|\int_a^b f-L(f,P)|<\frac{\epsilon}{3}$. Now,

$$\begin{aligned}\left|\int_a^b f-A\right| &= \left|\int_a^b f-L(f,P)+L(f,P)-S(f,P)+S(f,P)-A\right|\\ &\le \left|\int_a^b f-L(f,P)\right|+|L(f,P)-S(f,P)|+|S(f,P)-A|\\ &<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\ &=\epsilon.\end{aligned}$$

Hence $A=\int_a^b f$.

Theorem 3.2: *[]*

Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$. Then f is integrable if and only if there is a real number $A\in\mathbb{R}$ such that for $\epsilon>0$ there is a $\delta>0$ such that if P is any partition of $[a,b]$ with $\|P\|<\delta$ and if $S(f,P)$ is any Riemann Sum for f over P then $|S(f,P)-A|<\varepsilon$. If the condition is satisfied, then $A=\int_a^b f$.

Proof: $(\Rightarrow)$ Suppose f is integrable on $[a,b]$. Let $A=\int_a^b f$ and $\epsilon>0$ be given. There is a partition $P_\varepsilon=\{a=x_0,x_1,\cdots,x_N=b\}$ of $[a,b]$ such that $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\dfrac{\epsilon}{3}$. Let $M=\sup\limits_{x\in[a,b]}f(x)$ and $m=\inf\limits_{x\in[a,b]}f(x)$ and let $\delta=\dfrac{\epsilon}{3N(M-m)}$. ($N$ is the number of subintervals in P_ε) Let P be any partition of $[a,b]$ with $\|P\|<\delta$ and let $P^*=P\cup P_\varepsilon$ then $P_\varepsilon\subseteq P^*$. Now, since $P\subseteq P^*$, we have $U(f,P^*)\le U(f,P)$, $L(f,P)\le L(f,P^*)$, $U(f,P^*)\le U(f,P_\varepsilon)$, and $L(f,P_\varepsilon)\le L(f,P^*)$ we have $U(f,P)-U(f,P^*)\le N(M-m)\|P\|<N(M-m)\dfrac{\epsilon}{3N(M-m)}=\dfrac{\epsilon}{3}$ and $L(f,P^*)-L(f,P)\le N(M-m)\|P\|<N(M-m)\dfrac{\epsilon}{3N(M-m)}=\dfrac{\epsilon}{3}$ and $U(f,P^*)-L(f,P^*)\le U(f,P_\varepsilon)-L(f,P_\varepsilon)<\dfrac{\epsilon}{3}$.

Now, we have

$$L(f,P^*)-\frac{\epsilon}{3}<L(f,P)\le S(f,P)\le U(f,P)<U(f,P^*)+\frac{\epsilon}{3}<L(f,P^*)+\frac{2\epsilon}{3}$$

and

$$L(f,P^*)-\frac{\epsilon}{3}<L(f,P)\le\int_a^b f\le U(f,P)<U(f,P^*)+\frac{\epsilon}{3}<L(f,P^*)+\frac{2\epsilon}{3}$$

Hence

$$\left|S(f,P)-\int_a^b f\right|<\epsilon.$$

$(\Leftarrow)$ Suppose that the condition holds. Let $\epsilon>0$ be given. By the condition there is a $\delta>0$ such that if P is any partition of $[a,b]$ with $\|P\| < \delta$ and if $S(f,P)$ is any Riemann Sum for f over P then $|S(f,P) - A| < \epsilon$. Let Q be any partition of $[a,b]$ such that $P \subseteq Q$,then $\|Q\| \leq \|P\| < \delta$. Hence if $S(f,Q)$ is any Riemann Sum of f over Q, then $|S(f,Q) - A| < \epsilon$. Then by the previous theorem f is integrable and $A = \int\limits_a^b f$.

Theorem 3.3: []

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Then f is integrable if and only if for each sequence $\{P_n\}$ of partitions of $[a,b]$ with $\lim\limits_{n\to\infty} \|P_n\| = 0$, the sequence of Riemann sums $\{S(f,P_n)\}$ converges. If the condition is satisfied, then $\int\limits_a^b f = \lim\limits_{n\to\infty} S(f,P_n)$.

Proof: The proof is left as an exercise.

Example 3.2: Let

$$f(x) = \begin{cases} \dfrac{1}{q} & x = \dfrac{p}{q},\ (p,q) = 1 \\ 0 & \text{for irrational } x \text{ and } x = 0. \end{cases}$$

Prove that f is integrable on $[0,1]$ and find $\int\limits_0^1 f$.

Solution: For each $n \in \{2,3,5,7,11,13,17,\cdots\}$(the set of prime numbers) let $P_n = \{0, \frac{1}{n}, \cdots, \frac{n-1}{n}, 1\}$ then P_n is a partition of $[0,1]$ with $\|P_n\| = \frac{1}{n}$. For each $1 \leq k \leq n$ let $t_k = \frac{k}{n}$ then $t_k \in [\frac{k-1}{n}, \frac{k}{n}]$ and $(k,n) = 1$. Thus $f(t_k) = \frac{1}{n}$ for each $1 \leq k \leq n$.

Now $S(f,P_n) = \sum\limits_{k=1}^{n} f(t_k)\triangle x_k = \sum\limits_{k=1}^{n} \dfrac{1}{n}\dfrac{1}{n} = \sum\limits_{k=1}^{n} \dfrac{1}{n^2} = \dfrac{1}{n^2}.n = \dfrac{1}{n}$. Now, we have a sequence of partitions $\{P_n\}$ with $\lim\limits_{n\to\infty} \|P_n\| = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ and also we have the sequence of Riemann Sums $\{S(f,P_n)\}$ converges to 0 since $\lim\limits_{n\to\infty} S(f,P_n) = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Then f is integrable and $\int\limits_0^1 f = \lim\limits_{n\to\infty} S(f,P_n) = 0$.

