The Riemann Integral

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September 20, 2011

Definition 1.1: Let $[a, b] \subseteq \mathbb{R}$ be a closed and bounded interval.

i) A *Partition* of $[a, b]$ is a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < \ldots < x_n = b.$$

Note that the point of the partition divide $[a, b]$ into non-overlapping subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, and $[a, b] = \bigcup_{k=1}^{n} [x_{k-1}, x_k]$.

ii) The *norm* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is $\|P\| = \max\{x_k - x_{k-1} : k = 1, 2, \ldots, n\}$.

iii) A *refinement* of a partition $P = \{x_0, x_1, \ldots, x_n\}$ is a partition Q of $[a, b]$ such that $P \subseteq Q$.

Example 1.1: Let $[a, b] \subseteq \mathbb{R}$ be a closed and bounded interval and let $P_n = \left\{ x_k \mid x_k = a + \dfrac{k(b-a)}{n}, k = 0, 1, \ldots, n \right\}$, $\forall n \in \mathbb{N}$. Prove that for each $n \in \mathbb{N}$, P_n is a partition of $[a, b]$ and P_{2n} is a refinement of P_n.

Solution: We have $x_0 = a + \dfrac{0(b-a)}{n} = a$, and $x_n = a + \dfrac{n(b-a)}{n} = b$.

Now, if $1 \leq k \leq n$, we have

$$\begin{aligned} x_{k-1} &= a + \frac{(k-1)(b-a)}{n} \\ &< a + \frac{k(b-a)}{n} \\ &= x_k. \end{aligned}$$

Hence $a = x_0 < x_1 < \ldots < x_n = b$. Thus P_n is partition of $[a, b]$.

To see that $P_n \subset P_{2n}$: Let $0 \leq k \leq n$, then $0 \leq 2k \leq 2n$. Now, if $x_k \in P_n$, we have

$$\begin{aligned} x_k &= a + \frac{k(b-a)}{n} \\ &= a + \frac{2k(b-a)}{2n} \\ &= x_{2k} \in P_{2n}. \end{aligned}$$

Definition 1.2: Let $[a,b] \subseteq \mathbb{R}$ be a closed and bounded interval, $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$, and $f : [a,b] \to \mathbb{R}$ be bounded function. For $k = 1, 2, \ldots, n$, let $\triangle x_k = x_k - x_{k-1}$,
$M_k = M_k(f) = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}$, and $m_k = m_k(f) = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\}$.

i) The *upper Riemann sum* of f over P is the number

$$U(f,P) = \sum_{k=1}^{n} M_k \triangle x_k.$$

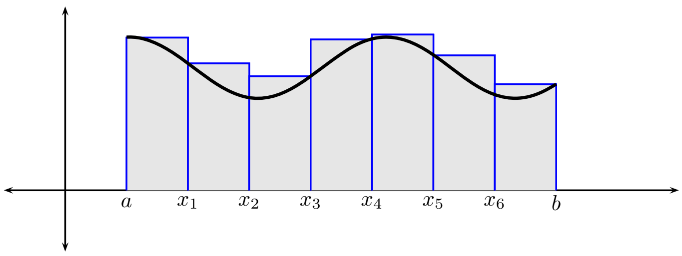

Figure 1: $U(f,P)$, an upper Riemann sum

ii) The *lower Riemann sum* of f over P is the number

$$L(f,P) = \sum_{k=1}^{n} m_k \triangle x_k.$$

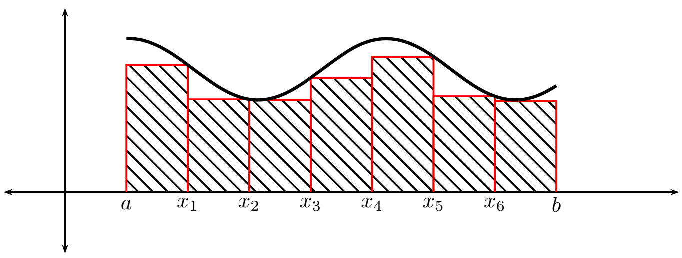

Figure 2: $L(f,P)$, a lower Riemann sum

Remark 1.1: Let $f : [a,b] \to \mathbb{R}$ be bounded function. Then $L(f,P) \leq U(f,P)$ for all partition P of $[a,b]$. ***Proof:*** Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since $m_k \leq M_k,\ \forall k = 1, 2, \ldots, n$, then

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k \triangle x_k \\ &\leq \sum_{k=1}^{n} M_k \triangle x_k \\ &= U(f,P). \end{aligned}$$

Hence $L(f,P) \leq U(f,P)$ for all partition P of $[a,b]$.

Example 1.2: Let $f : [0,1] \to \mathbb{R}$ given by $f(x) = x$, and for $n \in \mathbb{N}$, let $P_n = \left\{ x_k \mid x_k = \dfrac{k}{n}, k = 0, 1, \dots, n \right\}$ be a partition of $[0,1]$. Find $U(f, P_n)$ and $L(f, P_n)$.

Solution:

Since f is an increasing function we have $m_k = \frac{k-1}{n}$, $M_k = \frac{k}{n}$, and $\triangle x_k = \frac{1}{n}$, $k = 1, 2, \dots, n$. Thus

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} M_k \triangle x_k \\
&= \sum_{k=1}^{n} \left[\frac{k}{n}\right]\left[\frac{1}{n}\right] \\
&= \frac{1}{n}\left[\sum_{k=1}^{n} \frac{k}{n}\right] \\
&= \frac{1}{n}\left[\frac{1}{n}\sum_{k=1}^{n} k\right], \quad \text{we use } \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \\
&= \frac{1}{n}\left[\frac{1}{\cancel{n}}\frac{\cancel{n}(n+1)}{2}\right] \\
&= \frac{1}{n}\left[\frac{n+1}{2}\right] \\
&= \left[\frac{n+1}{2n}\right] \\
&= \frac{1}{2}\left[1 + \frac{1}{n}\right].
\end{aligned}$$

and

$$\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{n} \left[\frac{k-1}{n}\right]\left[\frac{1}{n}\right] \\
&= \frac{1}{n}\left[\sum_{k=1}^{n} \frac{k-1}{n}\right] \\
&= \frac{1}{n}\left[\frac{1}{n}\sum_{k=1}^{n} (k-1)\right], \quad \text{we use } \sum_{k=1}^{n} (k-1) = \frac{(n-1)((n+1)-1)}{2} = \frac{n(n-1)}{2} \\
&= \frac{1}{n}\left[\frac{1}{\cancel{n}}\frac{\cancel{n}(n-1)}{2}\right] \\
&= \frac{1}{n}\left[\frac{n-1}{2}\right] \\
&= \left[\frac{n-1}{2n}\right] \\
&= \frac{1}{2}\left[1 - \frac{1}{n}\right].
\end{aligned}$$

Note 1.1: Note that for any set A we have $\inf A \le \sup A$ and if $A \subseteq B$, then

$$\inf B \le \inf A \le \sup A \le \sup B.$$

Lemma 1.1: If P^* is a refinement of the partition P of $[a,b]$, then for any bounded function f define on $[a,b]$,

$$L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P).$$

Proof: Suppose $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$ and $P^* = \{x^*\} \cup P$ such that $x_{i-1} < x^* < x_i$ for some $1 \le i \le n$. Since $[x_{i-1}, x_i] = [x_{i-1}, x^*] \cup [x^*, x_i]$, then $m_i \le \inf\{f(x) \mid x \in [x_{i-1}, x^*]\}$ and $m_i \le \inf\{f(x) \mid x \in [x^*, x_i]\}$.

$$\begin{aligned}
L(f,P) &= \sum_{k=1}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{i-1} m_k \triangle x_k + m_i \triangle x_i + \sum_{k=i}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{i-1} m_k \triangle x_k + m_i (x_i - x_{i-1}) + \sum_{k=i}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{i-1} m_k \triangle x_k + m_i (x_i - x^* + x^* - x_{i-1}) + \sum_{k=i}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{i-1} m_k \triangle x_k + m_i (x_i - x^*) + m_i (x^* - x_{i-1}) + \sum_{k=i}^{n} m_k \triangle x_k \\
&\le \sum_{k=1}^{i-1} m_k \triangle x_k + [\inf\{f(x) \mid x \in [x^*, x_i]\}]\,(x_i - x^*) + [\inf\{f(x) \mid x \in [x_{i-1}, x^*]\}]\,(x^* - x_{i-1}) + \sum_{k=i}^{n} m_k \triangle x_k \\
&= L(f,P^*).
\end{aligned}$$

Hence $L(f,P) \le L(f,P^*)$.

Now, if $P^* = \{x_1^*, x_2^*, \dots, x_l^*\} \cup P$. Let $P_1 = \{x_1^*\} \cup P$, and $P_k = \{x_k^*\} \cup P_{k-1},\ k = 2, 3, \dots, l$. Thus we have a set of partitions of $[a,b]$, $P_1, P_2 < \dots, P_l$ such that $P_1 \subset P_2 \subset \dots \subset P_l = P^*$ and each partition is obtained from the preceding one by adding exactly one point. Then

$$L(f,P) \le L(f,P_1) \le L(f,P_2) \le \dots \le L(f,P_{l-1}) \le L(f,P_l) = L(f,P^*).$$

Similarly one can show $U(f,P^*) \le U(f,P)$. Therefore $L(f,P) \le L(f,P^*) \le U(f,P^*) \le U(f,P)$.

Remark 1.2: Let P be any partition of the interval $[a,b]$. Let $P^* = \{x^*\} \cup P$ and $P_k^* = \{x_1^*, x_2^*, \dots, x_k^*\} \cup P$ If $f : [a,b] \to \mathbb{R}$ be bounded function. Let $M = \sup\{f(x) \mid x \in [a,b]\}$, and $m = \inf\{f(x) \mid x \in [a,b]\}$. Then $L(f,P^*) - L(f,P) \le (M - m)\|P\|$, $U(f,P) - U(f,P^*) \le (M - m)\|P\|$, $L(f,P_k^*) - L(f,P) \le k(M - m)\|P\|$, and $U(f,P) - U(f,P_k^*) \le k(M - m)\|P\|$.

Proof: Suppose $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$ and $P^* = \{x^*\} \cup P$ such that $x_{i-1} < x^* < x_i$ for some $1 \le i \le n$. Since $[x_{i-1}, x_i] = [x_{i-1}, x^*] \cup [x^*, x_i]$, then $m \le m_i \le \inf\{f(x) \mid x \in [x_{i-1}, x^*]\}$ and
$m \le m_i \le \inf\{f(x) \mid x \in [x^*, x_i]\}$. Also $m \le \sup\{f(x) \mid x \in [x_{i-1}, x^*]\} \le M_i \le M$ and
$m \le \sup\{f(x) \mid x \in [x^*, x_i]\} \le M_i \le M$.

Hence

$$\begin{aligned} L(f,P^*)-L(f,P) &= \inf\{f(x)\mid x\in[x_{i-1},x^*]\}(x^*-x_{i-1})+\inf\{f(x)\mid x\in[x^*,x_i]\}(x_i-x^*)-m_i(x_i-x_{i-1})\\ &\le M(x^*-x_{i-1})+M(x_i-x^*)-m(x_i-x_{i-1})\\ &= M(x^*-x_{i-1}+x_i-x^*)-m(x_i-x_{i-1})\\ &\le (M-m)\|P\|. \end{aligned}$$

Also

$$\begin{aligned} U(f,P)-L(f,P^*) &= M_i(x_i-x_{i-1})-\sup\{f(x)\mid x\in[x_{i-1},x^*]\}(x^*-x_{i-1})-\sup\{f(x)\mid x\in[x^*,x_i]\}(x_i-x^*)\\ &\le M(x_i-x_{i-1})-m(x_i-x^*)-m(x_i-x_{i-1})\\ &= M(x_i-x_{i-1})-m(x^*-x_{i-1}+x_i-x^*)\\ &\le (M-m)\|P\|. \end{aligned}$$

By induction we can prove that $L(f,P_k^*)-L(f,P)\le k(M-m)\|P\|$, and $U(f,P)-U(f,P_k^*)\le k(M-m)\|P\|$. ■

Definition 1.3: Let $f:[a,b]\to\mathbb{R}$ be bounded function.

i) We define the *upper Riemann integral* of f on $[a,b]$ as

$$\begin{aligned} \overline{\int_a^b} f &= \inf\{U(f,P)\mid P \text{ a partitions of } [a,b]\}\\ &= \inf_P U(f,P). \end{aligned}$$

ii) and we define the *lower Riemann integral* of f on $[a,b]$ as

$$\begin{aligned} \underline{\int_a^b} f &= \sup\{L(f,P)\mid P \text{ a partitions of } [a,b]\}\\ &= \sup_P L(f,P). \end{aligned}$$

iii) f is called *Riemann integrable* on $[a,b]$ if $\overline{\int_a^b} f=\underline{\int_a^b} f$ and we define $\int_a^b f(x)\,dx$ to be the common value.

Example 1.3: Let $[a,b]\subseteq\mathbb{R}$ be a closed and bounded interval an let $f(x)=c$, for some $c\in\mathbb{R}$. Prove that f is integrable on $[a,b]$ and that $\int_a^b c\,dx=c(b-a)$

Solution:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$. Then $m_k = M_k = c,\ k = 1, 2, \ldots, n$. Thus

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M_k \triangle x_k \\ &= \sum_{k=1}^{n} c(x_k - x_{k-1}) \\ &= c\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= c\,[(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \ldots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})] \\ &= c(b - a) \end{aligned}$$

and

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} m_k \triangle x_k \\ &= \sum_{k=1}^{n} c(x_k - x_{k-1}) \\ &= c\sum_{k=1}^{n}(x_k - x_{k-1}) \\ &= c\,[(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \ldots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})] \\ &= c(b - a). \end{aligned}$$

Hence $\overline{\int_a^b} f = \underline{\int_a^b} f = c(b - a)$. Therefore $\int_a^b c\ dx = c(b - a)$.

Lemma 1.2: If P_1 and P_2 are any two partitions of $[a, b]$, then for any bounded function f define on $[a, b]$, then $L(f, P_1) \le U(f, P_2)$.

Proof: Note that if $P^* = P_1 \cup P_2$, then P^* is a refinement of both P_1 and P_2. Then from Lemma 1 $L(f, P_1) \le L(f, P^*) \le U(f, P^*) \le U(f, P_1)$, and $L(f, P_2) \le L(f, P^*) \le U(f, P^*) \le U(f, P_2)$. Hence $L(f, P_1) \le L(f, P^*) \le U(f, P^*) \le U(f, P_2)$. Therefore $L(f, P_1) \le U(f, P_2)$.

Lemma 1.3: For any bounded function f define on $[a, b]$, the upper Riemann integral and the lower Riemann integral exist, and $\underline{\int_a^b} f \le \overline{\int_a^b} f$.

Proof: By Lemma 2, $L(f, P_1) \le U(f, P_2)$ for all partitions P_1 and P_2 of $[a, b]$. Now, $U(f, P_2)$ is an upper bound for the set $\{L(f, P) \mid P \text{ a partitions of } [a, b]\}$, then
$\underline{\int_a^b} f = \sup\{L(f, P) \mid P \text{ a partitions of } [a, b]\} \le U(f, P_2)$. Since P_2 is an arbitrary partition of $[a, b]$, then $\underline{\int_a^b} f$ is a lower bound for the set $\{U(f, P) \mid P \text{ a partitions of } [a, b]\}$.
Thus $\underline{\int_a^b} f \le \inf\{U(f, P) \mid P \text{ a partitions of } [a, b]\} = \overline{\int_a^b} f$. Hence $\underline{\int_a^b} f \le \overline{\int_a^b} f$.

Note 1.2: Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$. For any partition P of $[a, b]$ we have

$$L(f, P) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f, P).$$

Example 1.4: Let $[a,b] \subseteq \mathbb{R}$ be a closed and bounded interval an let $f(x) = x$. Prove that f is integrable on $[a,b]$ and that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$

Solution:

Let P_n be the partition of $[a,b]$ into n subintervals given by $P_n = \left\{ x_k \mid x_k = a + \frac{k(b-a)}{n}, k = 0,1,\dots,n \right\}$. Since f is an increasing function we have $m_k = a + \frac{(k-1)(b-a)}{n}$, $M_k = a + \frac{k(b-a)}{n}$, and $\triangle x_k = \frac{b-a}{n}$, $k = 1,2,\dots,n$. Thus

$$
\begin{aligned}
U(f,P_n) &= \sum_{k=1}^{n} M_k \triangle x_k \\
&= \sum_{k=1}^{n} \left[a + \frac{k(b-a)}{n} \right] \left[\frac{b-a}{n} \right] \\
&= \frac{b-a}{n} \left[\sum_{k=1}^{n} a + \sum_{k=1}^{n} \frac{k(b-a)}{n} \right] \\
&= \frac{b-a}{n} \left[\sum_{k=1}^{n} a + \frac{b-a}{n} \sum_{k=1}^{n} k \right], \quad \text{we use } \sum_{k=1}^{n} a = na \text{ and } \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \\
&= \frac{b-a}{n} \left[na + \frac{(b-a)}{\cancel{n}} \frac{\cancel{n}(n+1)}{2} \right] \\
&= \frac{b-a}{n} \left[\frac{2na}{2} + \frac{(b-a)(n+1)}{2} \right] \\
&= \frac{b-a}{n} \left[\frac{2na - (n+1)a + b(n+1)}{2} \right] \\
&= \frac{b-a}{n} \left[\frac{2na - na - a + bn + b}{2} \right] \\
&= \frac{b-a}{n} \left[\frac{n(b+a) + (b-a)}{2} \right] \\
&= \left[\frac{n(b+a)(b-a) + (b-a)(b-a)}{2n} \right] \\
&= \left[\frac{\cancel{n}(b^2-a^2)}{2\cancel{n}} + \frac{(b-a)^2}{2n} \right] \\
&= \frac{b^2-a^2}{2} + \frac{(b-a)^2}{2n}.
\end{aligned}
$$

and

$$\begin{aligned}
L(f,P_n) &= \sum_{k=1}^{n} m_k \triangle x_k \\
&= \sum_{k=1}^{n}\left[a+\frac{(k-1)(b-a)}{n}\right]\left[\frac{b-a}{n}\right] \\
&= \frac{b-a}{n}\left[\sum_{k=1}^{n} a + \sum_{k=1}^{n}\frac{(k-1)(b-a)}{n}\right] \\
&= \frac{b-a}{n}\left[\sum_{k=1}^{n} a + \frac{b-a}{n}\sum_{k=1}^{n}(k-1)\right], \quad \text{we use } \sum_{k=1}^{n} a = na \text{ and } \sum_{k=1}^{n}(k-1) = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)n}{2} \\
&= \frac{b-a}{n}\left[na + \frac{(b-a)}{\cancel{n}}\frac{\cancel{n}(n-1)}{2}\right] \\
&= \frac{b-a}{n}\left[\frac{2na}{2} + \frac{(b-a)(n-1)}{2}\right] \\
&= \frac{b-a}{n}\left[\frac{2na-(n-1)a+b(n-1)}{2}\right] \\
&= \frac{b-a}{n}\left[\frac{2na-na+a+bn-b}{2}\right] \\
&= \frac{b-a}{n}\left[\frac{n(b+a)+(-b+a)}{2}\right] \\
&= \frac{b-a}{n}\left[\frac{n(b+a)-(b-a)}{2}\right] \\
&= \left[\frac{n(b+a)(b-a)-(b-a)(b-a)}{2n}\right] \\
&= \left[\frac{\cancel{n}(b^2-a^2)}{2\cancel{n}} - \frac{(b-a)^2}{2n}\right] \\
&= \frac{b^2-a^2}{2} - \frac{(b-a)^2}{2n}.
\end{aligned}$$

Now, since $\{U(f,P_n) \mid n \in \mathbb{N}\} \subset \{U(f,P) \mid P \text{ a partitions of } [a,b]\}$, then

$$\frac{b^2-a^2}{2} = \inf\left\{\frac{b^2-a^2}{2} + \frac{(b-a)^2}{2n} \mid n \in \mathbb{N}\right\} = \inf\{U(f,P_n) \mid n \in \mathbb{N}\} \geq \inf\{U(f,P) \mid P \text{ a partitions of } [a,b]\} = \overline{\int_a^b} x.$$

Thus $\overline{\int_a^b} x \leq \frac{b^2-a^2}{2}$.

Also, since $\{L(f,P_n) \mid n \in \mathbb{N}\} \subset \{L(f,P) \mid P \text{ a partitions of } [a,b]\}$, then

$$\frac{b^2-a^2}{2} = \sup\left\{\frac{b^2-a^2}{2} - \frac{(b-a)^2}{2n} \mid n \in \mathbb{N}\right\} = \sup\{L(f,P_n) \mid n \in \mathbb{N}\} \leq \sup\{L(f,P) \mid P \text{ a partitions of } [a,b]\} = \underline{\int_a^b} x.$$

Thus $\frac{b^2-a^2}{2} \leq \underline{\int_a^b} x$.

Therefore $\frac{b^2-a^2}{2} \leq \underline{\int_a^b} x \leq \overline{\int_a^b} x \leq \frac{b^2-a^2}{2}$. Thus $\underline{\int_a^b} x = \overline{\int_a^b} x = \frac{b^2-a^2}{2}$.

Theorem 1.1: ***[The Riemann Integrability Criterion I:]***

Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for $\varepsilon > 0$ there is a partition P_ε such that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

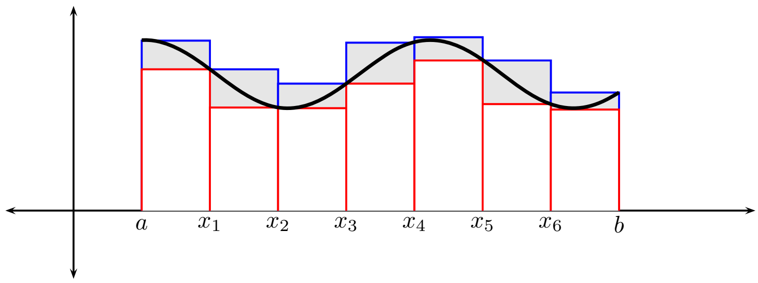

Figure 3: $U(f,P)-L(f,P)$

Proof: ($\Rightarrow$) Suppose that f is integrable, then $\inf_P U(f,P) = \overline{\int_a^b} f = \underline{\int_a^b} f = \sup_P L(f,P)$. Let $\varepsilon > 0$ be given. Since $\overline{\int_a^b} f + \frac{\varepsilon}{2}$ is not a lower bound of the set $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$, there is a partition P_1 of $[a,b]$ such that

$$\overline{\int_a^b} f < U(f,P_1) < \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

Similarly, since $\underline{\int_a^b} f - \frac{\varepsilon}{2}$ is not an upper bound of the set $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$, there is a partition P_2 of $[a,b]$ such that

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < L(f,P_2) < \underline{\int_a^b} f. \qquad \left(-\underline{\int_a^b} f < -L(f,P_2) < -\underline{\int_a^b} f + \frac{\varepsilon}{2}\right)$$

Let $P_\varepsilon = P_1 \cup P_2$, then P_ε is a refinement of both P_1 and P_2. Hence By Lemma 1, we have

$$L(f,P_2) \le L(f,P_\varepsilon) \text{ and } U(f,P_\varepsilon) \le U(f,P_1).$$

Now,

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < U(f,P_1) - L(f,P_2) < \overline{\int_a^b} f + \frac{\varepsilon}{2} - \left[\underline{\int_a^b} f - \frac{\varepsilon}{2}\right] = \overline{\int_a^b} f + \frac{\varepsilon}{2} - \underline{\int_a^b} f + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$) Suppose that for $\varepsilon > 0$ there is a partition P_ε such that

$$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

Now, let $\varepsilon > 0$ be given. Since

$$L(f,P_\varepsilon) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(f,P_\varepsilon),$$

then

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

Hence $\overline{\int_a^b} f = \underline{\int_a^b} f$. Thus f is integrable.

Corollary 1.1: Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$. If $\{P_n : n \in \mathbb{N}\}$ is a sequence of partitions of $[a,b]$ such that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0,$$

then f is integrable and

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

Proof: Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$, then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow U(f, P_n) - L(f, P_n) = |U(f, P_n) - L(f, P_n)| < \varepsilon$. Hence f is integrable.
Thus $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$. Therefore using Note 1 we have $L(f, P_n) \leq \int_a^b f \leq U(f, P_n)$. Now,

$$\text{if } n > N \Rightarrow \left| U(f, P_n) - \int_a^b f \right| = U(f, P_n) - \int_a^b f < U(f, P_n) - L(f, P_n) < \varepsilon.$$

Thus

$$\lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Also

$$\text{if } n > N \Rightarrow \left| \int_a^b f - L(f, P_n) \right| = \int_a^b f - L(f, P_n) < U(f, P_n) - L(f, P_n) < \varepsilon.$$

Hence

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f.$$

Example 1.5: Show that $\int_1^2 (3x + 1)\, dx = \frac{11}{2}$.

Solution: Let P_n be the partition of $[1, 2]$ into n subintervals given by $P_n = \left\{ x_k \mid x_k = 1 + \dfrac{k(2-1)}{n}, k = 0, 1, \ldots, n \right\}$.
Let $f(x) = 3x+1$. In each subinterval $[x_{k-1}, x_k] = \left[1 + \dfrac{k-1}{n}, 1 + \dfrac{k}{n}\right]$ the function f is increasing. Hence its maximum will be at $x = 1 + \dfrac{k}{n}$ and its minimum will be at $x = 1 + \dfrac{k-1}{n}$. Thus

$$m_k = 3\left(1 + \frac{(k-1)}{n}\right) + 1 = 3 + \frac{3(k-1)}{n} + 1 = 4 + \frac{3(k-1)}{n},$$

$$M_k = 3\left(1 + \frac{k}{n}\right) + 1 = 3 + \frac{3k}{n} + 1 = 4 + \frac{3k}{n}, \text{ and}$$

$$\triangle x_k = \frac{1}{n}, k = 1, 2, \ldots, n.$$

Thus

$$\begin{aligned} U(f,P_n) &= \sum_{k=1}^{n} M_k \triangle x_k \\ &= \sum_{k=1}^{n}\left[4+\frac{3k}{n}\right]\left[\frac{1}{n}\right] \\ &= \frac{1}{n}\left[\sum_{k=1}^{n}4+\sum_{k=1}^{n}\frac{3k}{n}\right] \\ &= \frac{1}{n}\left[\sum_{k=1}^{n}4+\frac{3}{n}\sum_{k=1}^{n}k\right] \\ &= \frac{1}{n}\left[4n+\frac{3}{\cancel{n}}\frac{\cancel{n}(n+1)}{2}\right] \\ &= \frac{1}{n}\left[\frac{11n+3}{2}\right] \\ &= \left[\frac{11n+3}{2n}\right] \\ &= \frac{11}{2}+\frac{3}{2n}. \end{aligned}$$

and

$$\begin{aligned} L(f,P_n) &= \sum_{k=1}^{n} m_k \triangle x_k \\ &= \sum_{k=1}^{n}\left[4+\frac{3(k-1)}{n}\right]\left[\frac{1}{n}\right] \\ &= \frac{1}{n}\left[\sum_{k=1}^{n}4+\sum_{k=1}^{n}\frac{3(k-1))}{n}\right] \\ &= \frac{1}{n}\left[\sum_{k=1}^{n}4+\frac{3}{n}\sum_{k=1}^{n}(k-1)\right], \quad \text{we use } \sum_{k=1}^{n}a = na \text{ and } \sum_{k=1}^{n}(k-1)=\frac{(n-1)((n-1)+1)}{2}=\frac{(n-1)n}{2} \\ &= \frac{1}{n}\left[4n+\frac{3}{\cancel{n}}\frac{\cancel{n}(n-1)}{2}\right] \\ &= \frac{1}{n}\left[\frac{2(4n)}{2}+\frac{3(n-1)}{2}\right] \\ &= \frac{1}{n}\left[\frac{8n+3n-3}{2}\right] \\ &= \frac{1}{n}\left[\frac{11n-3}{2}\right] \\ &= \left[\frac{11n-3}{2n}\right] \\ &= \left[\frac{11\cancel{n}}{2\cancel{n}}-\frac{3}{2n}\right] \\ &= \frac{11}{2}-\frac{3}{2n}. \end{aligned}$$

Now,

$$U(f,P_n)-L(f,P_n)=\left[\frac{11}{2}+\frac{3}{2n}\right]-\left[\frac{11}{2}-\frac{3}{2n}\right]=\frac{6}{2n}=\frac{3}{n}, \text{ hence } \lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0.$$

Therefore

$$\int_1^2 (3x+1)\,dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty}\left[\frac{11}{2}+\frac{3}{2n}\right] = \frac{11}{2}.$$

Theorem 1.2: *[The Riemann Integrability Criterion II:]*
Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for $\varepsilon>0$ there is a $\delta=\delta(\varepsilon)>0$ such that if P is any partition with $\|P\|<\delta$, then $U(f,P)-L(f,P)<\varepsilon$.

Proof: ($\Rightarrow$) Suppose that f is integrable. Let $\varepsilon>0$ be given, then by Theorem 0.1 there is a partition $P_\varepsilon=\{a=x_0,x_1,\ldots,x_N=b\}$ such that $U(f,P_\varepsilon)-L(f,P_\varepsilon)<\frac{\varepsilon}{2}$. We want to show that there is a $\delta>0$ such that if P is any partition of $[a,b]$ with $\|P\|<\delta$, then $U(f,P)-L(f,P)<\varepsilon$. Let $\delta=\frac{\varepsilon}{4(M-m)N}$. If P is any partition of $[a,b]$ with $\|P\|<\delta$. Let $Q=P\cup P_\varepsilon$, then by Remark2 we have

$$L(f,Q)-L(f,P)\le N(M-m)\|P\|<N(M-m)\delta=N(M-m)\frac{\varepsilon}{4N(M-m)}=\frac{\varepsilon}{4}.$$

Thus $L(f,Q)-L(f,P)<\frac{\varepsilon}{4}$.

Also

$$U(f,P)-U(f,Q)\le N(M-m)\|P\|<N(M-m)\delta=N(M-m)\frac{\varepsilon}{4N(M-m)}=\frac{\varepsilon}{4}.$$

Hence $U(f,P)-U(f,Q)<\frac{\varepsilon}{4}$. Now,

$$\begin{aligned}
\text{since } P_\varepsilon\subseteq Q &\Rightarrow L(f,P_\varepsilon)\le L(f,Q)\\
&\Rightarrow L(f,P_\varepsilon)-L(f,P)\le L(f,Q)-L(f,P)<\frac{\varepsilon}{4} \qquad (1)\\
\text{Also, since } P_\varepsilon\subseteq Q &\Rightarrow U(f,Q)\le U(f,P_\varepsilon)\\
&\Rightarrow -U(f,P_\varepsilon)\le -U(f,Q)\\
&\Rightarrow U(f,P)-U(f,P_\varepsilon)\le U(f,P)-U(f,Q)<\frac{\varepsilon}{4}. \qquad (2)
\end{aligned}$$

Now,

$$\begin{aligned}
L(f,P_\varepsilon)-L(f,P) &< \frac{\varepsilon}{4}\\
U(f,P)-U(f,P_\varepsilon) &< \frac{\varepsilon}{4} \qquad \text{adding the two inequities}\\
U(f,P)-L(f,P)-(U(f,P_\varepsilon)-L(f,P_\varepsilon)) &< \frac{\varepsilon}{4}+\frac{\varepsilon}{4}\\
U(f,P)-L(f,P) &< U(f,P_\varepsilon)-L(f,P_\varepsilon)+\frac{\varepsilon}{2}\\
U(f,P)-L(f,P) &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

($\Leftarrow$) Suppose that for $\varepsilon>0$ there is a $\delta>0$ such that if P is any partition with $\|P\|<\delta$, then $U(f,P)-L(f,P)<\varepsilon$. Now, let $\varepsilon>0$ be given. Choose a partition P such that $\|P\|<\delta$, then $U(f,P)-L(f,P)<\varepsilon$. Hence f is integrable by Theorem 0.1.