Monotone Sequences

Dr.Hamed Al-Sulami

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3.1 Monotone Sequences

Definition 3.1: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of a real numbers.

(1) We say $\{x_n\}_{n=1}^{\infty}$ is ***increasing*** if it satisfies the inequalities $x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$.

(2) We say $\{x_n\}_{n=1}^{\infty}$ is ***decreasing*** if it satisfies the inequalities $x_1 \geq x_2 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$.

(3) We say $\{x_n\}_{n=1}^{\infty}$ is ***monotone*** if it is either increasing or decreasing.

Note 3.1: If $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence and if $n, k \in \mathbb{N}$ such that $n > k, \Rightarrow x_k \leq x_n$. If $\{x_n\}_{n=1}^{\infty}$ is an decreasing sequence and if $n, k \in \mathbb{N}$ such that $n > k, \Rightarrow x_n \leq x_k$.

Example 3.1:

(a) $\{\frac{1}{n}\}$ is decreasing sequence since $\frac{1}{n} > \frac{1}{n+1}$ $\forall\ n \in \mathbb{N}$.

(b) $\{1 - \frac{1}{n^2}\}$ is increasing sequence since $1 - \frac{1}{n^2} < 1 - \frac{1}{(n-1)^2}$.

(c) $\{(-1)^n\}$ is not monotone.

Theorem 3.1: ***[Monotone Convergence Theorem -MCT]***

A monotone sequence of real numbers is convergent if and only if it is bounded. Moreover:

(a) If $\{x_n\}$ is bounded above increasing sequence and $x = \sup\{x_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} x_n = x$.

(b) If $\{y_n\}$ is bounded below decreasing sequence and $y = \inf\{y_n : n \in \mathbb{N}\}$, then $\lim_{n\to\infty} y_n = y$.

Proof:

(a) Since $\{x_n\}$ is bounded above, then $\sup\{x_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$. Let $x = \sup\{x_n : n \in \mathbb{N}\}$.
We want to show that $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$ be given. Since $x - \epsilon$ is not an upper bound of $\{x_n : n \in \mathbb{N}\}$, then there exist $N \in \mathbb{N}$ such that $x - \epsilon < x_N$. Now, if $n > N$, since $\{x_n\}$ is increasing sequence, then $x_N \leq x_n$. If $n > N \Rightarrow x - \epsilon < x_N \leq x_n \leq x < x + \epsilon$. Hence, if $n > N \Rightarrow x - \epsilon < x_n < x + \epsilon$. Thus, if $n > N \Rightarrow |x_n - x| < \epsilon$. Therefore $\lim_{n\to\infty} x_n = x$.

(b) Since $\{y_n\}$ is bounded below, then $\inf\{y_n : n \in \mathbb{N}\}$ exists in $\mathbb{R}$. Let $y = \inf\{y_n : n \in \mathbb{N}\}$. ***We want to show that*** $\lim_{n\to\infty} y_n = y$. Let $\epsilon > 0$ be given. Since $y + \epsilon$ is not a lower bound of $\{y_n : n \in \mathbb{N}\}$, then there exist $N \in \mathbb{N}$ such that $y_N < y + \epsilon$. Now, if $n > N$, since $\{y_n\}$ is decreasing sequence, then $y_n \leq y_N$. If

$n > N \Rightarrow y - \epsilon < y \leq y_n \leq y_N < y + \epsilon$. Hence, if $n > N \Rightarrow y - \epsilon < y_n < y + \epsilon$. Thus, if $n > N \Rightarrow |y_n - y| < \epsilon$. Therefore $\lim\limits_{n\to\infty} y_n = y$.

Example 3.2: Let $x_1 = 1$ and $x_{n+1} = 1 - \sqrt{3 - x_n}$, for all $n \in \mathbb{N}$.

(a) Prove $-1 \leq x_{n+1} \leq x_n \leq 1$, for all $n \in \mathbb{N}$.

(b) Prove that $\lim\limits_{n\to\infty} x_n = -1$.

Solution:

(a) We will use mathematical induction to show $-1 \leq x_{n+1} \leq x_n \leq 1$. Suppose it is true for n. Thus $-1 \leq x_{n+1} \leq x_n \leq 1$, and we will prove it for $n + 1$.

Now, we have

$$\begin{aligned}
-1 \leq x_{n+1} &\Leftrightarrow -x_{n+1} \leq 1 \\
&\Leftrightarrow 3 - x_{n+1} \leq 4 \\
&\Leftrightarrow \sqrt{3 - x_{n+1}} \leq \sqrt{4} \\
&\Leftrightarrow -\sqrt{4} \leq -\sqrt{3 - x_{n+1}} \\
&\Leftrightarrow 1 - 2 \leq 1 - \sqrt{3 - x_{n+1}} \\
&\Leftrightarrow -1 \leq x_{n+2}. \qquad (1)
\end{aligned}$$

Also,we have

$$\begin{aligned}
x_{n+1} \leq x_n &\Leftrightarrow -x_n \leq -x_{n+1} \\
&\Leftrightarrow 3 - x_n \leq 3 - x_{n+1} \\
&\Leftrightarrow \sqrt{3 - x_n} \leq \sqrt{3 - x_{n+1}} \\
&\Leftrightarrow -\sqrt{3 - x_{n+1}} \leq -\sqrt{3 - x_n} \\
&\Leftrightarrow 1 - \sqrt{3 - x_{n+1}} \leq 1 - \sqrt{3 - x_n} \\
&\Leftrightarrow x_{n+2} \leq x_{n+1}. \qquad (2)
\end{aligned}$$

Finally, we have

$$\begin{aligned}
x_n \leq 1 &\Leftrightarrow -1 \leq -x_n \\
&\Leftrightarrow 3 - 1 \leq 3 - x_n \\
&\Leftrightarrow \sqrt{3 - 1} \leq \sqrt{3 - x_n} \\
&\Leftrightarrow -\sqrt{3 - x_n} \leq -\sqrt{2} \\
&\Leftrightarrow 1 - \sqrt{3 - x_n} \leq 1 - \sqrt{2} < 1 \\
&\Leftrightarrow x_{n+1} \leq 1. \qquad (3)
\end{aligned}$$

From (1),(2),and (3) we get $-1 \leq x_{n+2} \leq x_{n+1} \leq 1$. Thus $-1 \leq x_{n+1} \leq x_n \leq 1$, for all $n \in \mathbb{N}$.

(b) Since $\{x_n\}$ is decreasing bounded sequence, then by MCT $\{x_n\}$ is convergent. Also, since $-1 \le x_n \le 1$, then $-1 \le \lim_{n\to\infty} x_n \le 1$. Now, let $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_{n+1} = x$ also. Since $x_{n+1} = 1 - \sqrt{3 - x_n}$, then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(1 - \sqrt{3 - x_n}) = 1 - \sqrt{3 - \lim_{n\to\infty} x_n}$. Hence $x = 1 - \sqrt{3 - x}$. Thus $(x-1)^2 = 3 - x$. Hence $x^2 - 2x + 1 = 3 - x$. Thus $x^2 - x - 2 = 0$. Hence $(x+1)(x-2) = 0$. Thus $x = -1$, or $x = 2$. But since $-1 \le x \le 1$, then $x \ne 2$. Therefore $\lim_{n\to\infty} x_n = -1$.

Example 3.3: Let $x_1 = 2$ and $x_{n+1} = \sqrt{2x_n + 3}$, for all $n \in \mathbb{N}$.

(a) Prove $2 \le x_n \le x_{n+1} \le 3$.

(b) Prove that $\lim_{n\to\infty} x_n = 3$.

Solution:

(a) We will use mathematical induction to show $2 \le x_n \le x_{n+1} \le 3$. Suppose it is true for n. Thus $2 \le x_n \le x_{n+1} \le 3$, and we will prove it for $n+1$. Now, we have

$$\begin{aligned} 2 \le x_n &\Leftrightarrow 4 \le 2x_n \\ &\Leftrightarrow 4 + 3 \le 2x_n + 3 \\ &\Leftrightarrow \sqrt{7} \le \sqrt{2x_n + 3} \\ &\Leftrightarrow 2 < \sqrt{7} \le \sqrt{2x_n + 3} \\ &\Leftrightarrow 2 < x_{n+1}. \end{aligned} \tag{1}$$

Also,we have

$$\begin{aligned} x_n \le x_{n+1} &\Leftrightarrow 2x_n \le 2x_{n+1} \\ &\Leftrightarrow 2x_n + 3 \le 2x_{n+1} + 3 \\ &\Leftrightarrow \sqrt{2x_n + 3} \le \sqrt{2x_{n+1} + 3} \\ &\Leftrightarrow x_{n+1} \le x_{n+2}. \end{aligned} \tag{2}$$

Finally, we have

$$\begin{aligned} x_{n+1} \le 3 &\Leftrightarrow 2x_{n+1} \le 6 \\ &\Leftrightarrow 2x_{n+1} + 3 \le 6 + 3 \\ &\Leftrightarrow \sqrt{2x_{n+1} + 3} \le \sqrt{9} \\ &\Leftrightarrow \sqrt{2x_{n+1} + 3} \le 3 \\ &\Leftrightarrow x_{n+2} \le 3. \end{aligned} \tag{3}$$

From (1),(2),and (3) we get $2 \le x_{n+1} \le x_{n+2} \le 3$. Thus $2 \le x_n \le x_{n+1} \le 3$, for all $n \in \mathbb{N}$.

(b) Since $\{x_n\}$ is increasing bounded sequence, then by MCT $\{x_n\}$ is convergent. Also, since $2 \le x_n \le 3$, then $2 \le \lim_{n\to\infty} x_n \le 3$. Now, let $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_{n+1} = x$ Also since $x_{n+1} = \sqrt{2x_n + 3}$, then $\lim_{n\to\infty} x_{n+1} =$

$\lim_{n\to\infty}(\sqrt{2x_n+3}) = \sqrt{2\lim_{n\to\infty} x_n + 3}$. Hence $x = \sqrt{2x+3}$. Thus $(x)^2 = 2x+3$. Hence $x^2 - 2x - 3 = 0$. Thus $(x+1)(x-3) = 0$. Thus $x = -1$, or $x = 3$. But since $2 \le x \le 3$, then $x \ne -1$. Therefore $\lim_{n\to\infty} x_n = 3$.

Example 3.4: Let $a > 0$, $x_1 > 0$, and $x_{n+1} = \dfrac{x_n + \frac{a}{x_n}}{2}$, for all $n \in \mathbb{N}$ and $n \ge 2$.

(a) Prove $\sqrt{a} \le x_{n+1} \le x_n$.

(b) Prove that $\lim_{n\to\infty} x_n = \sqrt{a}$.

Solution:

(a) Note that $x_n > 0$ now, we have

$$\begin{aligned}
x_{n+1} = \frac{x_n + \frac{a}{x_n}}{2} &\Leftrightarrow 2x_{n+1} = x_n + \frac{a}{x_n} \\
&\Leftrightarrow 2x_{n+1}x_n = x_n^2 + a \\
&\Leftrightarrow -a = x_n^2 - 2x_n x_{n+1} \\
&\Leftrightarrow x_{n+1}^2 - a = x_n^2 - 2x_n x_{n+1} + x_{n+1}^2 = (x_n - x_{n+1})^2 \ge 0 \\
&\Leftrightarrow x_{n+1}^2 \ge a \\
&\Leftrightarrow \sqrt{a} \le x_{n+1}. \qquad (1)
\end{aligned}$$

Also,we have

$$\begin{aligned}
x_n - x_{n+1} &= x_n - \frac{x_n + \frac{a}{x_n}}{2} \\
&= \frac{2x_n}{2} - \frac{x_n + \frac{a}{x_n}}{2} \\
&= \frac{2x_n - x_n - \frac{a}{x_n}}{2} \\
&= \frac{x_n - \frac{a}{x_n}}{2} \\
&= \frac{x_n^2 - a}{2x_n} \ge 0 \\
&\Leftrightarrow x_n - x_{n+1} \ge 0 \\
&\Leftrightarrow x_{n+1} \le x_n. \qquad (2)
\end{aligned}$$

From (1), and (2) we get $\sqrt{a} \le x_{n+1} \le x_n$. Thus $\sqrt{a} \le x_{n+1} \le x_n$, for all $n \in \mathbb{N}$ such that $n \ge 2$.

(b) Since $\{x_n\}$ is decreasing bounded below sequence, then by MCT $\{x_n\}$ is convergent. Also, since $\sqrt{a} \le x_{n+1} \le x_2$, then $\sqrt{a} \le \lim_{n\to\infty} x_n \le x_2$. Now, let $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_{n+1} = x$ Also since $x_{n+1} = \dfrac{x_n + \frac{a}{x_n}}{2}$, then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(\dfrac{x_n + \frac{a}{x_n}}{2}) = \dfrac{x + \frac{a}{x}}{2}$. Hence $x = \dfrac{x + \frac{a}{x}}{2}$. Thus $x = \dfrac{x^2 + a}{2x}$. Hence $2x^2 = x^2 + a$. Thus $x^2 = a$. Thus $x = \pm\sqrt{a}$. But since $x_n > 0$, then $x = \lim_{n\to\infty} x_n \ge 0$. Therefore $\lim_{n\to\infty} x_n = \sqrt{a}$.

Example 3.5: Let $e_n = (1 + \dfrac{1}{n})^n$, for all $n \in \mathbb{N}$. Prove that $\{e_n\}$ is increasing and bounded.

Solution: We will use The Binomial Theorem to expand $(1+\frac{1}{n})^n$.

$$
\begin{aligned}
e_n = \left(1+\frac{1}{n}\right)^n &= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{n}\right)^i \\
&= \sum_{i=0}^{n}\binom{n}{i}\frac{1}{n^i} && \text{Note that: } \binom{n}{i}\frac{1}{n^i} = \frac{1}{i!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{i-1}{n}) < \frac{1}{i!} \\
&< \sum_{i=0}^{n}\frac{1}{i!} \\
&= 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} && \text{Note that: } \frac{1}{i!} \le \frac{1}{2^{i-1}} \text{ for all } i \ge 3 \\
&< 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} \\
&< 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} && \text{Note that: } 1+\frac{1}{2}+\ldots\frac{1}{2^{n-1}} = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2-\frac{1}{2^{n-1}} \\
&= 1+2-\frac{1}{2^{n-1}} \\
&< 3.
\end{aligned}
$$

Thus $2 = e_1 < e_n < 3$.Thus $\{e_n\}$ is bounded .

$$
\begin{aligned}
e_n = \left(1+\frac{1}{n}\right)^n &= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{1}{n}\right)^i \\
&= \sum_{i=0}^{n}\frac{1}{i!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{i-1}{n}) && \text{and} \\
e_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} &= \sum_{i=0}^{n+1}\frac{1}{i!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\ldots(1-\frac{i-1}{n+1}) \\
&\ge \sum_{i=0}^{n}\frac{1}{i!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\ldots(1-\frac{i-1}{n+1}) && \text{Note that: } 1-\frac{k}{n+1} > 1-\frac{k}{n} \quad \forall k \in \mathbb{N} \\
&> \sum_{i=0}^{n}\frac{1}{i!}(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{i-1}{n}) = e_n.
\end{aligned}
$$

Hence $e_n < e_{n+1}$. Therefore $\{e_n\}$ is increasing and bounded. Thus it is convergent.The limit of this sequence is the number e.

Example 3.6: Let $x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, for all $n \in \mathbb{N}$. Prove that $\{x_n\}$ is increasing and unbounded.

Solution:

$$
\begin{aligned}
x_{n+1} &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1} \\
&> 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = x_n
\end{aligned}
$$

Thus $\{s_n\}$ is increasing.

$$
\begin{aligned}
x_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\
&= 1 + \frac{n}{2}.
\end{aligned}
$$

Since $\{x_n\}$ is unbounded.Thus $\{x_n\}$ is divergent.