Math 311 all sections Fall 2013

You must solve question number one and any other two.

1. For the sequence $\{\frac{2n^2+3}{3n^2-n}\}_{n=1}^{\infty}$ find the limit and prove your answer using the ***definition***.

Solution:

$$\lim_{n\to\infty}\frac{2n^2+3}{3n^2-n}=\lim_{n\to\infty}\frac{\frac{2n^2}{n^2}+\frac{3}{n^2}}{\frac{3n^2}{n^2}-\frac{n}{n^2}}=\lim_{n\to\infty}\frac{2+\frac{3}{n^2}}{3-\frac{1}{n}}=\frac{2}{3}.$$

Discussion:

We want to show $\lim_{n\to\infty}\frac{2n^2+3}{3n^2-n}=\frac{2}{3}$.

$$\begin{aligned}
\left|\frac{2n^2+3}{3n^2-n}-\frac{2}{3}\right| &= \left|\frac{6n^2+9-6n^2+2n}{9n^2-3n}\right| \\
&= \frac{2n+9}{9n^2-3n} && \text{Note that: } 2n+9\le 2n+9n=11n \\
&\le \frac{11n}{9n^2-3n} && \text{Note that: } 9n^2-3n\ge 9n^2-3n^2 \Leftrightarrow \frac{1}{9n^2-3n}\le\frac{1}{9n^2-3n^2} \\
&\le \frac{11n}{6n^2}=\frac{11}{6n}. \\
\text{Now, let } \frac{11}{6n} &< \varepsilon \\
\Leftrightarrow n &> \frac{11}{6\varepsilon}.
\end{aligned}$$

Proof of $\lim_{n\to\infty}\frac{2n^2+3}{3n^2-n}=\frac{2}{3}$ **:**

Let $\varepsilon>0$ be given. Let $N\in\mathbb{N}$ such that $N>\frac{11}{6\varepsilon}$.

$$\begin{aligned}
\text{Now, if } n>N &\Rightarrow \frac{1}{n}<\frac{1}{N}<\frac{6\varepsilon}{11} \\
&\Rightarrow \frac{11}{6n}<\varepsilon \\
&\Rightarrow \left|\frac{2n^2+3}{3n^2-n}-\frac{2}{3}\right|<\frac{11}{6n}<\varepsilon \\
\text{Now, if } n>N &\Rightarrow \left|\frac{2n^2+3}{3n^2-n}-\frac{2}{3}\right|<\varepsilon. \\
\text{Therefore } &\lim_{n\to\infty}\frac{2n^2+3}{3n^2-n}=\frac{2}{3}.
\end{aligned}$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq C|x - y|$, for all $x, y \in \mathbb{R}$ and for some $C > 0$. Let $\{a_n\}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$.

(**a**) Prove that $\lim\limits_{n\to\infty} f(a_n) = f(a)$.

Solution:

Discussion:

Since $|f(x) - f(y)| \leq C|x - y|$, for all $x, y \in \mathbb{R}$, then $|f(a_n) - f(a)| \leq C|a_n - a|$, and we want $C|a_n - a| < \epsilon \Rightarrow |a_n - a| < \frac{\epsilon}{C}$.

Proof:

Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}$ then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |a_n - a| < \frac{\epsilon}{C}$. Now if $n > N \Rightarrow |f(a_n) - f(a)| \leq C|a_n - a| < C.\frac{\epsilon}{C} = \epsilon$. Hence $\lim\limits_{n\to\infty} f(a_n) = f(a)$.

(**b**) Is the sequence $\{\tan^{-1}(1 + \frac{1}{n})\}$ convergent? If it is convergent find the limit.

Solution:

From Calculus we know that $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|, \forall x, y \in \mathbb{R}$. Since $\lim\limits_{n\to\infty} [1 + \frac{1}{n}] = 1$, then $\{\tan^{-1}(1 + \frac{1}{n})\}$ is convergent and the $\lim\limits_{n\to\infty} \tan^{-1}(1 + \frac{1}{n}) = \tan^{-1} 1 = \frac{\pi}{4}$.

3. Let $\{x_n\}$ be a convergent sequence and $\{y_n\}$ is a divergent sequence.

(**a**) Prove that $\{x_n + y_n\}$ is divergent.

Solution:

Suppose that $\{x_n + y_n\}$ is convergent. Now since $\{x_n\}$ is convergent and $\{x_n + y_n\}$ is convergent, then difference between them is convergent. Since $y_n = x_n + y_n - x_n$ then $\{y_n\}$ is convergent. Contradiction, therefor $\{x_n + y_n\}$ is divergent.

(**b**) Is the sequence $\{(-1)^n + \frac{1}{n}\}$ convergent? If it is convergent find the limit.

Solution:

Since $\{(-1)^n\}$ is divergent and $\{\frac{1}{n}\}$ is convergent, then by **(a)** $\{(-1)^n + \frac{1}{n}\}$ is divergent.

4. Let $\{x_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}^+$. Let $m \in \mathbb{Z}$.

(a) Prove that $\lim\limits_{n\to\infty} x_n^m = x^m$.

Solution:

Discussion:

Note that if $m > 0$, we have $x_n^m - x^m = (x_n - x)\sum\limits_{k=1}^{m} x_n^{m-k}x^{k-1}$.

$|x_n^m - x^m| = |x_n - x|.|\sum\limits_{k=1}^{m} x_n^{m-k}x^{k-1}| \leq |x_n - x|(\sum\limits_{k=1}^{m} |x_n|^{m-k}|x|^{k-1})$. Now, we want to get raid of $|x_n|^{m-k}$ we do this by using the fact that $\{x_n\}$ is bounded because it is convergent. Hence there exist $M > 0$ such that $|x_n| \leq M$ for all n.

Hence $|x_n^m - x^m| = |x_n - x|.|\sum\limits_{k=1}^{m} x_n^{m-k}x^{k-1}| \leq |x_n - x|(\sum\limits_{k=1}^{m} |x_n|^{m-k}|x|^{k-1} \leq |x_n - x|(\sum\limits_{k=1}^{m} M^{m-k}|x|^{k-1})$. Let $\beta = \sum\limits_{k=1}^{m} M^{m-k}|x|^{k-1} \in \mathbb{R}^+$. Thus $|x_n^m - x^m| \leq \beta|x_n - x|$. We want $\beta|x_n - x| < \epsilon \Rightarrow |x_n - x| < \frac{\epsilon}{\beta}$.

Proof:

Case I: If $m > 0$

Let $\epsilon > 0$ be given. Since $\{x_n\}$ is convergent, then there exist $M > 0$ such that $|x_n| \leq M$ for all n.

Since $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}^+$, then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{\beta}$.

Now, if $n > N \Rightarrow |x_n^m - x^m| \leq \beta|x_n - x| < \beta.\frac{\epsilon}{\beta} = \epsilon$.

Hence $\lim\limits_{n\to\infty} x_n^m = x^m$.

Case II: If $m < 0$

Note that $-m > 0$ and $a^m = \frac{1}{a^{-m}} = \left(\frac{1}{a}\right)^{-m}$. Since $x_n, x > 0$. then $\lim\limits_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$ and by Case I $\lim\limits_{n\to\infty} \left(\frac{1}{x_n}\right)^{-m} = \left(\frac{1}{x}\right)^{-m}$. Now, $\lim\limits_{n\to\infty} x_n^m = \lim\limits_{n\to\infty} \frac{1}{x_n^{-m}} = \lim\limits_{n\to\infty} \left(\frac{1}{x_n}\right)^{-m} = \left(\frac{1}{x}\right)^{-m} = \frac{1}{x^{-m}} = x^m$.

Case III: If $m = 0$

$\lim\limits_{n\to\infty} x_n^m = \lim\limits_{n\to\infty} x_n^0 = \lim\limits_{n\to\infty} 1 = 1 = x^0 = x^m$.

(b) Prove that there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow \frac{x}{3} < x_n < 3x$.

Solution:

Discussion:

Since $x_n, x > 0$, and $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}^+$, then there exist $N_1 \in \mathbb{N}$ such that

$$
\begin{aligned}
\text{if } n > N_1 &\Rightarrow |x_n - x| < \epsilon' \\
&\Rightarrow -\epsilon' < x_n - x < \epsilon' \\
&\Rightarrow x - \epsilon' < x_n < x + \epsilon'
\end{aligned}
$$

We want $x + \epsilon' = 3x \Leftrightarrow \epsilon' = 2x$.

Since $x_n, x > 0$, and $\lim_{n\to\infty} x_n = x \in \mathbb{R}^+$, then there exist $N_2 \in \mathbb{N}$ such that

$$\begin{aligned}\text{if } n > N_2 &\Rightarrow |x_n - x| < \epsilon'' \\ &\Rightarrow -\epsilon'' < x_n - x < \epsilon'' \\ &\Rightarrow x - \epsilon'' < x_n < x + \epsilon''\end{aligned}$$

We want $x - \epsilon'' = \frac{x}{3} \Leftrightarrow \epsilon'' = \frac{2x}{3}$.

Proof:

Since $x_n, x > 0$, and $\lim_{n\to\infty} x_n = x \in \mathbb{R}^+$, then there exist $N_1 \in \mathbb{N}$ such that

$$\begin{aligned}\text{if } n > N_1 &\Rightarrow |x_n - x| < 2x \\ &\Rightarrow -2x < x_n - x < 2x \\ &\Rightarrow x - 2x < x_n < x + 2x \\ &\Rightarrow -x < x_n < 3x\end{aligned}$$

Hence there exist $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow x_n < 3x$.

Also since $x_n, x > 0$, and $\lim_{n\to\infty} x_n = x \in \mathbb{R}^+$, then there exist $N_2 \in \mathbb{N}$ such that

$$\begin{aligned}\text{if } n > N_2 &\Rightarrow |x_n - x| < \frac{2x}{3} \\ &\Rightarrow -\frac{2x}{3} < x_n - x < \frac{2x}{3} \\ &\Rightarrow x - \frac{2x}{3} < x_n < x + \frac{2x}{3} \\ &\Rightarrow \frac{x}{3} < x_n < \frac{5x}{3}\end{aligned}$$

Hence there exist $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow \frac{x}{3} < x_n$.

Let $N = \max\{N_1, N_2\}$, if $n > N$ then $n > N_1$ and $n > N_2$ and hence $n > N \Rightarrow \frac{x}{3} < x_n < 3x$.

5. Let $\{x_n\}$ be a sequence of real numbers.

(**a**) Prove that there exist a sequence $\{m_n\} \subset \mathbb{Z}$ such that $\lim\limits_{n\to\infty} [x_n - \frac{m_n}{n}] = 0$.

Solution:

Discussion:

Note that for $x \in \mathbb{R}, \exists! m \in \mathbb{Z} \ni m \leq x < m+1$. Now, for each $n \in \mathbb{N}$, since $nx_n \in \mathbb{R}$, then there exist $m_n \in \mathbb{Z}$ such that $m_n \leq nx_n < m_n + 1$. Hence $\frac{m_n}{n} \leq x_n < \frac{m_n}{n} + \frac{1}{n}$. Therefore for each $n \in \mathbb{N}$, since $nx_n \in \mathbb{R}$, then there exist $m_n \in \mathbb{Z}$ such that $0 = \frac{m_n}{n} - \frac{m_n}{n} \leq x_n - \frac{m_n}{n} < \frac{m_n}{n} + \frac{1}{n} - \frac{m_n}{n} + \frac{1}{n} = \frac{1}{n}$. Thus $0 \leq |x_n - \frac{m_n}{n}| = x_n - \frac{m_n}{n} < \frac{1}{n}$. We can choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

Proof:

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

$$
\begin{aligned}
\text{If } n > N &\Rightarrow \frac{1}{n} < \frac{1}{N} < \epsilon \\
&\Rightarrow |x_n - \frac{m_n}{n}| = x_n - \frac{m_n}{n} < \frac{1}{n} < \frac{1}{N} < \epsilon \\
\text{If } n > N &\Rightarrow |x_n - \frac{m_n}{n} - 0| = |x_n - \frac{m_n}{n}| < \epsilon
\end{aligned}
$$

Hence $\lim\limits_{n\to\infty} [x_n - \frac{m_n}{n}] = 0$.

(**b**) If $\lim\limits_{n\to\infty} x_n = x$, prove that $\lim\limits_{n\to\infty} \frac{m_n}{n} = x$.

Solution:

Since $\frac{m_n}{n} = x_n - [x_n - \frac{m_n}{n}]$ and since $\lim\limits_{n\to\infty} x_n = x$, and $\lim\limits_{n\to\infty} [x_n - \frac{m_n}{n}] = 0$.
Then $\lim\limits_{n\to\infty} \frac{m_n}{n} = \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} [x_n - \frac{m_n}{n}] = x - 0 = x$.

6. Let $\{y_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} y_n = 0$.

(**a**) Let $\{x_n\}$ be a sequence of real numbers and $x \in \mathbb{R}$. Suppose that there is $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| \leq y_n$. Prove that $\lim_{n\to\infty} x_n = x$.

Solution:

Discussion:

Since $y_n > 0$, and $\lim_{n\to\infty} y_n = 0$ then there exist $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow y_n = |y_n| = |y_n - 0| < \epsilon'$. Given that there is $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| \leq y_n$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$ and if $n > N \Rightarrow |x_n - x| \leq y_n < \epsilon'$. We want $\epsilon' = \epsilon$.

Proof:

Let $\epsilon > 0$ be given. Since $y_n > 0$, and $\lim_{n\to\infty} y_n = 0$ then there exist $N_2 \in \mathbb{N}$ such that if $n > N_2 \Rightarrow y_n = |y_n| = |y_n - 0| < \epsilon$. Given that there is $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| \leq y_n$. Let $N = \max\{N_1, N_2\} \in \mathbb{N}$ and if $n > N \Rightarrow |x_n - x| \leq y_n < \epsilon$. Hence $\lim_{n\to\infty} x_n = x$.

(**b**) Suppose that $\{x_n\}$ is bounded sequence. Prove that $\lim_{n\to\infty} x_n y_n = 0$.

Solution:

Discussion:

Since $\{x_n\}$, is bounded sequence then there exist $M > 0$ such that $|x_n| \leq M$ for all n. Since $y_n > 0$, and $\lim_{n\to\infty} y_n = 0$ then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow y_n = |y_n| = |y_n - 0| < \epsilon'$. Now, if $n > N \Rightarrow |x_n y_n - 0| = |x_n y_n| = |x_n| y_n < M y_n < M\epsilon'$. We want $M\epsilon' = \epsilon \Leftrightarrow \epsilon' = \frac{\epsilon}{M}$

Proof:

Let $\epsilon > 0$ be given. Since $y_n > 0$, and $\lim_{n\to\infty} y_n = 0$ then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow y_n = |y_n| = |y_n - 0| < \frac{\epsilon}{M}$. Now, if $n > N \Rightarrow |x_n y_n - 0| = |x_n y_n| = |x_n| y_n < M y_n < M\frac{\epsilon}{M} = \epsilon$. Hence $\lim_{n\to\infty} x_n y_n = 0$.

7. Let $x_1 = 1$ and $x_{n+1} = \sqrt{2x_n + 3}$, for $n \geq 2$

(**a**) Prove that $1 \leq x_n \leq 3$.

Solution: We use mathematical induction to prove this (you did this in Math251) Prove it for $n = 1$. Assume it is true for $n = k$ and use it to prove it for $n = k + 1$.

Since $1 \leq 1 = x_1 \leq 3$, then it is true for $n = 1$. Assume that $1 \leq x_k \leq 3$, prove that $1 \leq x_{k+1} \leq 3$.

$$
\begin{aligned}
1 \leq x_k \leq 3 &\Rightarrow 2 \leq 2x_k \leq 6 && \text{multiply all sides by 2.}\\
&\Rightarrow 1 < 2 + 3 \leq 2x_k + 3 \leq 6 + 3 = 9 && \text{add 3 to all sides.}\\
&\Rightarrow 1 = \sqrt{1} \leq \sqrt{2x_k + 3} \leq \sqrt{9} = 3 && \text{take the square root of all sides. Note } x_{k+1} = \sqrt{2x_k + 3}.\\
&\Rightarrow 1 \leq x_{k+1} \leq 3
\end{aligned}
$$

Hence $1 \leq x_n \leq 3$ and $\{x_n\}$ is bounded.

(**b**) Prove that $x_n \leq x_{n+1}$.

Solution: Since $1 \leq 5 \Rightarrow 1 = \sqrt{1} \leq \sqrt{5} \Rightarrow x_1 = 1 \leq \sqrt{5} = x_2$ then it is true for $n = 1$. Assume that $x_k \leq x_{k+1}$, use it to prove that $x_{k+1} \leq x_{k+2}$.

$$
\begin{aligned}
x_k \leq x_{k+1} &\Rightarrow 2x_k \leq 2x_{k+1} && \text{multiply all sides by 2.}\\
&\Rightarrow 2x_k + 3 \leq 2x_{k+1} + 3 && \text{add 3 to all sides.}\\
&\Rightarrow \sqrt{2x_k + 3} \leq \sqrt{2x_{k+1} + 3} && \text{take the square root of all sides. Note } x_{k+1} = \sqrt{2x_k + 3} \text{ and } x_{k+2} = \sqrt{2x_{k+1} + 3}\,.\\
&\Rightarrow x_{k+1} \leq x_{k+2}
\end{aligned}
$$

Hence $x_n \leq x_{n+1}$ and $\{x_n\}$ is increasing.

(**c**) Prove that $\{x_n\}$ is convergent and find its limit.

Solution: Since $\{x_n\}$ is bounded and $\{x_n\}$ is increasing, then by MCT $\{x_n\}$ is convergent. Let $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. Since $1 \leq x_n \leq 3 \Rightarrow 1 \leq x = \lim_{n\to\infty} x_n \leq 3$.

$$
\begin{aligned}
x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} &\Rightarrow x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{2x_n + 3} = \sqrt{2x + 3} && \text{use } x_{n+1} = \sqrt{2x_n + 3}.\\
&\Rightarrow x = \sqrt{2x + 3} && \text{square both sides.}\\
&\Rightarrow x^2 = 2x + 3 && \text{make it a zero equation .}\\
&\Rightarrow x^2 - 2x - 3 = 0 && \text{two integers their sum is } -2 \text{ and their product is } -3.\\
&\Rightarrow (x - 3)(x + 1) = 0 && \text{the two integers are } -3 \text{ and } 1\\
&\Rightarrow x = 3 \text{ or } x = -1 && \text{since } 1 \leq x \leq 3, \text{ then } x \neq -1.
\end{aligned}
$$

Hence $\lim_{n\to\infty} x_n = 3$.

8. Let $x_1 = \frac{3}{2}$ and $x_{n+1} = 2 - \frac{1}{x_n}$, for $n \geq 2$

(a) Prove that $1 < x_n < 2$.

Solution:

Since $1 < \frac{3}{2} = x_1 < 2$, then it is true for $n = 1$. Assume that $1 < x_k < 2$, prove that $1 < x_{k+1} < 2$.

$$\begin{aligned}
1 < x_k < 2 &\Rightarrow 1 > \frac{1}{x_k} > \frac{1}{2} && \text{Use } 0 < a < b \Leftrightarrow \tfrac{1}{a} > \tfrac{1}{b}. \\
&\Rightarrow -1 < -\frac{1}{x_k} < -\frac{1}{2} && \text{Multiply all sides by } -1. \\
&\Rightarrow 1 = 2 - 1 < 2 - \frac{1}{x_k} < 2 - \frac{1}{2} = \frac{3}{2} < 2 && \text{add 2 to all sides. Note } x_{k+1} = 2 - \tfrac{1}{x_k}. \\
&\Rightarrow 1 < x_{k+1} < 2
\end{aligned}$$

Hence $1 < x_n < 2$ and $\{x_n\}$ is bounded.

(b) Prove that $x_n > x_{n+1}$.

Solution: Since $x_1 = \frac{3}{2} = 1\frac{1}{2} > 1\frac{1}{3} = \frac{4}{3} = 2 - \frac{1}{\frac{3}{2}} = x_2$ then it is true for $n = 1$. Assume that $x_k > x_{k+1}$, use it to prove that $x_{k+1} > x_{k+2}$.

$$\begin{aligned}
x_k > x_{k+1} &\Rightarrow \frac{1}{x_k} < \frac{1}{x_{k+1}} && \text{Use } 0 < a < b \Leftrightarrow \tfrac{1}{a} > \tfrac{1}{b}.. \\
&\Rightarrow -\frac{1}{x_k} > -\frac{1}{x_{k+1}} && \text{Multiply all sides by } -1. \\
&\Rightarrow 2 - \frac{1}{x_k} > 2 - \frac{1}{x_{k+1}} && \text{add 2 to all sides. Note } x_{k+1} = 2 - \tfrac{1}{x_k} \text{ and } x_{k+2} = 2 - \tfrac{1}{x_{k+1}}\ . \\
&\Rightarrow x_{k+1} > x_{k+2}
\end{aligned}$$

Hence $x_n > x_{n+1}$ and $\{x_n\}$ is decreasing.

(c) Prove that $\{x_n\}$ is convergent and find its limit.

Solution: Since $\{x_n\}$ is bounded and $\{x_n\}$ is decreasing, then by MCT $\{x_n\}$ is convergent. Let $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. Since $1 < x_n < 2 \Rightarrow 1 \leq x = \lim_{n\to\infty} x_n \leq 2$.

$$\begin{aligned}
x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} &\Rightarrow x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} 2 - \frac{1}{x_n} = 2 - \frac{1}{x} && \text{use } x_{n+1} = 2 - \tfrac{1}{x_n}. \\
&\Rightarrow x = 2 - \frac{1}{x} && \text{Multiply both sides by } x. \\
&\Rightarrow x^2 = 2x - 1 && \text{make it a zero equation .} \\
&\Rightarrow x^2 - 2x + 1 = 0 && \text{two integers their sum is } -2 \text{ and their product is 1.} \\
&\Rightarrow (x-1)(x-1) = 0 && \text{the two integers are } -1 \text{ and } -1 \\
&\Rightarrow x = 1 \text{ or } x = 1
\end{aligned}$$

Hence $\lim_{n\to\infty} x_n = 1$.