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ANALYTIC DETERMINATION OF THE POLE OF A GALAXY CLUSTER

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ABSTRACT

In this paper, an analytic method is introduced to determine the pole of a cluster of galaxies. The method depends on a constrained optimization technique. A general computational algorithm for the procedure was established to find exact solutions for the equations involved. A typical application of this process to the Coma Cluster is also presented. Excellent agreement is found between the result of the present method and the empirical method which has been recently developed by the author. This provides strong evidence of the reliability of the methods.

Subject headings: galaxies: clustering

I. INTRODUCTION

In a recent work by the author (Bukhari 1987, hereafter Paper I) the basic parameters that describe a cluster of galaxies were determined empirically by an iterative method. In the present paper, a pure analytic procedure is introduced to specify the equatorial coordinates of the pole of a galaxy cluster. The latter method depends primarily on a minimizing technique which is subject to side conditions. The data for the Coma Cluster that are used here for the application of this method are exactly identical to those given in Paper I. In § II the formulation of the analytic method is discussed in detail, while the general computational algorithm is developed in § III. As an example, the process as applied to the Coma Cluster is given in § IV, together with a comparison between the results of both the analytic and empirical approaches.

II. FORMULATION OF THE METHOD

Assume a coordinate system having its origin in the Sun and oriented (see Fig. 1) as follows: the x-axis is toward the point γ with coordinates $\alpha = 0^h$, $\delta = 0°$; the y-axis is toward the point B with coordinates $\alpha = 6^h$, $\delta = 0°$; and the z-axis is toward the North Celestial Pole N for a given epoch.

The normal equation of a plane is

$$lx + my + nz - p = 0 , \qquad (2.1)$$

where p is the distance of the plane from the origin and l, m, and n are the direction cosines of the perpendicular to the plane drawn from the origin. They fulfill the equation

$$l^2 + m^2 + n^2 = 1 . \qquad (2.2)$$

From the spherical triangle $AP\gamma$ (Fig. 1) we have

$$l = \cos P\gamma = \cos \delta_p \cos \alpha_p , \qquad (2.3)$$

where α_p, δ_p are the equatorial coordinates of the point toward which the perpendicular is directed. It is also obvious from Figure 1 that

$$n = \cos PN = \sin \delta_p . \qquad (2.4)$$

Substituting equations (2.3) and (2.4) for l and n in equation (2.2) we get

$$m = \cos \delta_p \sin \alpha_p . \qquad (2.5)$$

Let α_i, δ_i be the equatorial coordinates of one of the investigated N galaxies. Assume that they are all at unit distance from the Sun; thus we have

$$x_i = \cos \delta_i \cos \alpha_i , \qquad y_i = \cos \delta_i \sin \alpha_i ,$$
$$z_i = \sin \delta_i , \qquad i = 1, 2, \ldots, N . \qquad (2.6)$$

The distance of any galaxy i from a plane drawn through the origin is

$$D_i = lx_i + my_i + nz_i , \qquad i = 1, 2, \ldots, N . \qquad (2.7)$$

The sum of the squared distances from the plane is given by

$$\sum_i D_i^2 = \sum_i (lx_i + my_i + nz_i)^2 = f_1(l, m, n) , \qquad (2.8)$$

which obviously is a function of the direction cosines of the perpendicular—l, m, and n—i.e., a function of the plane orientation.

The problem is how to find a set of values of l, m, and n which minimizes expression (2.8). Now, we cannot simply equate with zero the partial derivatives of equation (2.8) with respect to l, m, and n, as these variables are subject to the side condition (2.2), which we will write as

$$f_2(l, m, n) = l^2 + m^2 + n^2 - 1 = 0 . \qquad (2.9)$$

Hence, the problem is reduced to a typical extreme value problem with side conditions. The solution is most easily found by Lagrange's method of indeterminate multipliers. For that purpose multiply equation (2.9) by $-\lambda$ and add the result to equation (2.8). This gives

$$F(l, m, n) = f_1(l, m, n) - \lambda f_2(l, m, n) , \qquad (2.10)$$

or, explicitly,

$$F(l, m, n) = (a_{11} - \lambda)l^2 + (a_{22} - \lambda)m^2 + (a_{33} - \lambda)n^2 + 2(a_{12} lm + a_{13} ln + a_{23} mn) + \lambda , \qquad (2.11)$$

where

$$a_{11} = \sum_i x_i^2 , \qquad a_{12} = \sum_i x_i y_i , \qquad a_{13} = \sum_i x_i z_i ,$$
$$a_{22} = \sum_i y_i^2 , \qquad a_{23} = \sum_i y_i z_i , \qquad a_{33} = \sum_i z_i^2 . \qquad (2.12)$$

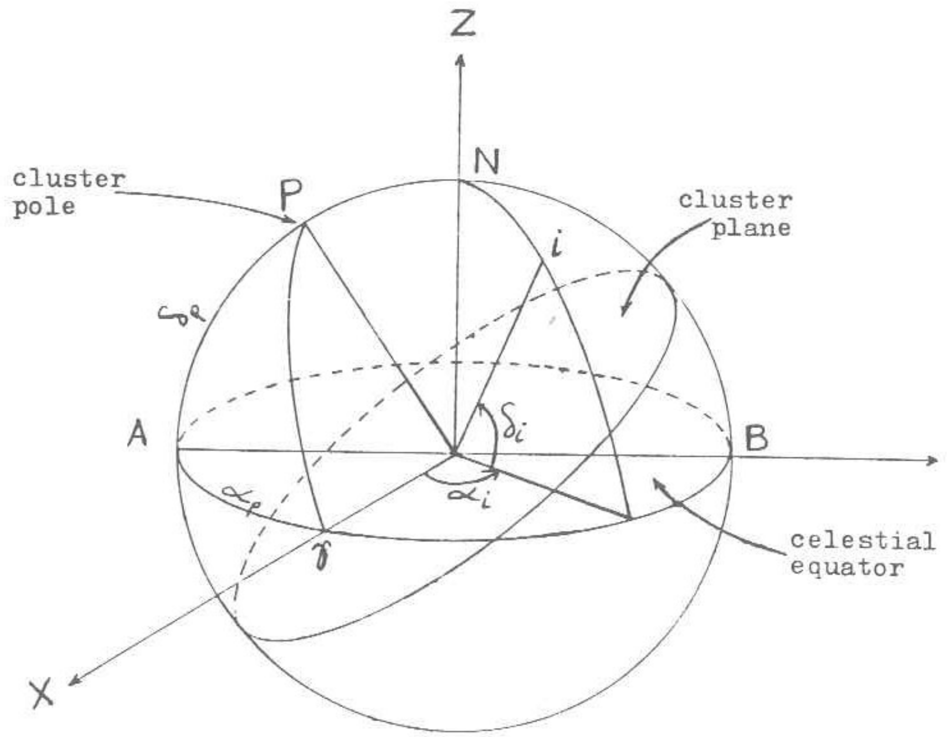

FIG. 1.—The geometry of the analytic method

The necessary conditions for an extremum are

$$\frac{\partial F}{\partial l} = 0 , \quad \frac{\partial F}{\partial m} = 0 , \quad \frac{\partial F}{\partial n} = 0 . \tag{2.13}$$

That is,

$$\begin{aligned} (a_{11} - \lambda)l + a_{12} m + a_{13} n &= 0 , \\ a_{12} l + (a_{22} - \lambda)m + a_{23} n &= 0 , \\ a_{13} l + a_{23} m + (a_{33} - \lambda)n &= 0 . \end{aligned} \tag{2.14}$$

The equations (2.14) represent a system of homogeneous linear equations which could be written as

$$\mathbf{A}x = \lambda x , \tag{2.15}$$

where **A** is the symmetric matrix

$$\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} \tag{2.16}$$

and x is the vector of the direction cosines

$$x = \begin{bmatrix} l \\ m \\ n \end{bmatrix} . \tag{2.17}$$

Apart from the trivial solution $l = m = n = 0$, the nontrivial solution of the system (2.14) is only obtainable if

$$\mathbf{D}(\lambda) = \mathbf{A} - \lambda I = 0 . \tag{2.18}$$

This is the characteristic equation of the matrix **A**, which could be written in the form

$$\mathbf{D}(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{12} & a_{22} - \lambda & a_{23} \\ a_{13} & a_{23} & a_{33} - \lambda \end{vmatrix} = 0 . \tag{2.19}$$

By developing the determinant, equation (2.19) yields

$$\mathbf{D}(\lambda) = \lambda^3 + k_1 \lambda^2 + k_2 \lambda + k_3 = 0 , \tag{2.20}$$

where the coefficients k_1, k_2, and k_3 are given by

$$\begin{aligned} k_1 &= -(a_{11} + a_{22} + a_{33}) , \\ k_2 &= a_{11} a_{22} + a_{11} a_{33} + a_{22} a_{33} - (a_{12}^2 + a_{13}^2 + a_{23}^2) , \\ k_3 &= a_{12}^2 a_{33} + a_{13}^2 a_{22} + a_{23}^2 a_{11} - a_{11} a_{22} a_{33} - 2a_{12} a_{13} a_{23} . \end{aligned} \tag{2.21}$$

Equation (2.20) is a cubic from which Lagrange's multiplier λ can be evaluated. It is remarkable to note that, since the matrix **A** is symmetric, its eigenvalues are all real.

Equations (2.3), (2.4), (2.5), (2.12), (2.20), and (2.21) represent the basic analytic formulation of the method.

III. COMPUTATIONAL ALGORITHM

In this section we shall develop a general computational algorithm to determine the roots of equation (2.20) and, consequently, to determine the equatorial coordinates of the cluster pole:

1. Calculate q and r where

$$q = \frac{1}{3} k_2 - \frac{1}{9} k_1^2 , \qquad r = \frac{1}{6} (k_1 k_2 - 3k_3) - \frac{1}{27} k_1^3 , \tag{3.1}$$

where k_1, k_2, and k_3 are evaluated from equations (2.21). Check that $q^3 + r^2 < 0$ to confirm that the values of λ are all real.

2. Obtain ρ, x, and ϕ using the formulae

$$\rho = \sqrt{-q^3}\,, \qquad x = \rho^2 - r^2\,, \qquad \phi = \tan^{-1}(\sqrt{x}/r) \quad (3.2)$$

3. Find the required roots from the expressions

$$\begin{aligned} \lambda_1 &= 2\rho^{1/3}\cos(\phi/3) - k_1/3\,, \\ \lambda_2 &= -\rho^{1/3}[\cos(\phi/3) + \sqrt{3}\sin(\phi/3)] - k_1/3\,, \\ \lambda_3 &= -\rho^{1/3}[\cos(\phi/3) - \sqrt{3}\sin(\phi/3)] - k_1/3\,. \end{aligned} \quad (3.3)$$

4. Having got the values of Lagrange's multiplier we can proceed to the solution of the linear system of equations (2.14) for the direction cosines

$$x^T = \{l, m, n\}\,.$$

Let λ_i be one of the three roots; then equations (2.14) become

$$\begin{aligned} (a_{11} - \lambda_i)l_i + a_{12}m_i + a_{13}n_i &= 0\,, \\ a_{12}l_i + (a_{22} - \lambda_i)m_i + a_{23}n_i &= 0\,, \\ a_{13}l_i + a_{23}m_i + (a_{33} - \lambda_i)n_i &= 0\,, \qquad i = 1, 2, 3\,. \end{aligned} \quad (3.4)$$

This is a system of homogeneous linear equations for the corresponding values l_i, m_i, and n_i of the direction cosines.

Consider the last two equations of equations (3.4) divided by n_i; then

$$\begin{aligned} a_{12}\xi + (a_{22} - \lambda_i)\eta &= -a_{23}\,, \\ a_{13}\xi + a_{23}\eta &= -(a_{33} - \lambda_i)\,, \qquad i = 1, 2, 3\,, \end{aligned} \quad (3.5)$$

where

$$\xi = l_i/n_i\,, \qquad \eta = m_i/n_i \quad (3.6)$$

Hence, the solution for the system (3.5) is

$$\xi = G_{11}^{(i)}/G_{13}^{(i)}\,, \qquad \eta = G_{12}^{(i)}/G_{13}^{(i)}\,,$$

or

$$l_i = (G_{11}^{(i)}/G_{13}^{(i)})n_i\,, \qquad m_i = (G_{12}^{(i)}/G_{13}^{(i)})n_i\,, \qquad i = 1, 2, 3 \quad (3.7)$$

where

$$\begin{aligned} G_{11}^{(i)} &= (a_{22} - \lambda_i)(a_{33} - \lambda_i) - a_{23}^2\,, \\ G_{12}^{(i)} &= -a_{12}(a_{33} - \lambda_i) + a_{23}a_{13}\,, \\ G_{13}^{(i)} &= a_{12}a_{23} - a_{13}(a_{22} - \lambda_i)\,, \qquad i = 1, 2, 3\,. \end{aligned} \quad (3.8)$$

Now the direction cosines are subject to the condition

$$l_i^2 + m_i^2 + n_i^2 = 1\,, \qquad i = 1, 2, 3\,. \quad (3.9)$$

Substituting l_i and n_i from equation (3.7) into equation (3.9), it could easily be found that

$$n_i = G_{13}^{(i)}/R_i\,, \quad (3.10)$$

where

$$R_i^2 = [G_{11}^{(i)}]^2 + [G_{12}^{(i)}]^2 + [G_{13}^{(i)}]^2\,. \quad (3.11)$$

By using equations (3.7) and (3.10), we get the following forms for the required direction cosines

$$l_i = G_{11}^{(i)}/R_i\,, \quad m_i = G_{12}^{(i)}/R_i\,, \quad n_i = G_{13}^{(i)}/R_i\,, \quad i = 1, 2, 3\,, \quad (3.12)$$

where $G_{11}^{(i)}$, $G_{12}^{(i)}$, $G_{13}^{(i)}$, and R_i are given by equations (3.8) and (3.11), respectively.

5. Eventually, by means of equations (2.3), (2.4), and (2.5) the equatorial coordinates of the galaxy cluster pole are computed from

$$\alpha_{p_i} = \tan^{-1}(m_i/l_i)\,, \quad \delta_{p_i} = \sin^{-1}(n_i)\,, \quad i = 1, 2, 3\,. \quad (3.13)$$

IV. RESULTS

When the method described in §§ II and III was applied to the brightest half of the galaxies of the Coma Cluster, it yielded three pairs of equatorial coordinates for the pole. The values of α_p and δ_p corresponding to the minimum value of $F(l, m, n)$ are

$$\alpha_p = 11^h 25^m 46^s.97$$
$$\delta_p = 61°47'51''.15$$

The average equatorial coordinates of the 138 bright galaxies which form the Coma plane (determined by the empirical process of Paper I) are

$$\alpha_{p_n} = 12^h 57^m 10^s.13$$
$$\delta_{p_n} = 28°11'14''.89$$

It is obvious that $\delta_{p_n} = 90 - \delta_p$, which is the result expected from the geometry of the analytic method as indicated in Figure 1. This significant consistence between the results of the empirical and the analytical methods provides clear evidence that both are dependable. The values of α_{p_n} and δ_{p_n} are also in full agreement with 1950 equatorial coordinates of the Coma Cluster presented by Abell (1958). It is noticeable that the equatorial coordinates of the cluster pole calculated by the algorithm of § III are exact in the sense that no approximation was made.

In conclusion, the analytic method which was introduced to determine the pole of galaxy cluster depends on a constrained optimization technique. A general computational algorithm of the process has been established to find exact solutions of the equations involved. A typical application of this procedure to the Coma Cluster was also demonstrated. Excellent agreement was found between the results of both the present method and the empirical one which has been recently developed by the author (Paper I). This is a measure of the reliability of the methods.

It will be interesting to apply this approach to other clusters of galaxies, particularly those of linear and flat (L, F) types in the Rood-Sastry (Rood-Sastry 1971) classification system.

REFERENCES

Abell, G. O. 1958, *Ap. J. Suppl.*, **3**, 211.
Bahcall, N. A. 1977, *Ann. Rev. Ap.*, **15**, 505.
Bukhari, F. A. 1987, *Ap. J.*, **331**, 000.
Carter, D., and Metcalfe, N. 1980, *M.N.R.A.S.*, **191**, 325.
Gregory, S. A., Thompson, L. A., and Tifft, W. G. 1981, *Ap. J.*, **243**, 411.
Hawley, D. L., and Peebles, P. J. E. 1975, *Ap. J.*, **80**, 477.
MacGillivray, H. T., Dodd, R. T., McNally, B. V., and Corwin, H. G. 1982, *M.N.R.A.S.*, **198**, 605.
MacGillivray, H. T., and Stobie, R. S. 1985, *Vistas Astr.*, Vol. **27**, No. 4, p. 433.
Rood, H. J., and Sastry, G. N. 1971, *Pub. Astr. Soc. Pacific*, **83**, 313.
Trumpler, R. J., and Weaver, H. F. 1953, *Statistical Astronomy* (Mineola, NY: Dover).

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