بسم الله الرحمن الرحيم

- الرجاء كتابة اسمُك و رقمك الجامعي في المكان المخصص أسفل هذه الصفحة.
- تأكد من حصولك على جميع الأسئلة.
- الرجاء أغلق الجوال و عدم أستخدامه خلال الاختبار.
- يحتوي هذا الإختبار على **سبعة** أسئلة والمطلوب الأجابة على **ستة** منها، علماً بأن السؤالين **الأول و الثالث** أجباريان.
- الزمن المحدد لهذا الإختبار 120 دقيقة.
- بالتوفيق إن شاء الله.

Question:	1	2	3	4	5	6	7	Total
Points:	25	11	30	11	11	11	11	110
Score:								

Name : ______________________________

Student's I.N.: ______________________________

1. Let $x \in \mathbb{R}$, $\{x_n\}$ be a sequence of real numbers, and $E \subseteq \mathbb{R}$. State the definition of the following:

   (a) E is an *open set* of $\mathbb{R}$. [5]

   Solution:
   A set E of real numbers is said to be ***open set*** if, for each $x \in E$ there is a number $\delta > 0$ such that $(x - \delta, x + \delta) \subset E$.

   (b) The *Bolzano-Weierstrass Theorem.* [5]

   Solution:
   Let $\{x_n\}_{n=1}^{\infty}$ be bounded sequence of a real numbers, then $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.

   (c) The sequence $\{x_n\}$ is *Cauchy* . [5]

   Solution:
   A sequence $\{x_n\}$ of real numbers is said to be ***Cauchy sequence*** if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N \Rightarrow |x_n - x_m| < \epsilon$.

   (d) x is a *limit point* of E. [5]

   Solution:
   We say x is a ***limit point*** of E if for every $\delta > 0$, $(x - \delta, x + \delta) \cap (E \setminus \{x\}) \neq \emptyset$.

   (e) The *Monotone Convergence Theorem.* [5]

   Solution:
   A monotone sequence of real numbers is convergent if and only if it is bounded. Moreover:

   (a) If $\{x_n\}$ is bounded above increasing sequence and $x = \sup\{x_n : n \in \mathbb{N}\}$, then $\lim\limits_{n\to\infty} x_n = x$.

   (b) If $\{y_n\}$ is bounded below decreasing sequence and $y = \inf\{y_n : n \in \mathbb{N}\}$, then $\lim\limits_{n\to\infty} y_n = y$.

2. Let $x_1 = 1.1$ and $x_{n+1} = \sqrt{3x_n + 4}$, for all $n \in \mathbb{N}$.

(a) Prove $1 < x_n < 4$, for all $n \in \mathbb{N}$. [4]

Solution:
We will use mathematical induction to show $1 < x_n < 2$. Suppose it is true for n. Thus $1 < x_n < 2$, and we will prove it for $n+1$. Now, we have

$$\begin{aligned} 1 < x_n < 4 &\Leftrightarrow 3(1) < 3x_n < 3(4) \\ &\Leftrightarrow 3 < 3x_n < 12 \\ &\Leftrightarrow 3+4 < 3x_n + 4 < 12+4 \\ &\Leftrightarrow \sqrt{7} < \sqrt{3x_n+4} < \sqrt{16} \\ &\Leftrightarrow 1 < \sqrt{7} < \sqrt{3x_n+4} < \sqrt{16} = 4 \qquad \Leftrightarrow 1 < x_{n+1} < 4. \end{aligned} \tag{1}$$

Hence $1 < x_n < 4$, for all $n \in \mathbb{N}$.

(b) Prove $x_n \le x_{n+1}$, for all $n \in \mathbb{N}$. [4]

Solution:
We will use mathematical induction to show $x_n \le x_{n+1}$. Suppose it is true for n. Thus $x_n \le x_{n+1}$, and we will prove it for $n+1$. Now, we have

$$\begin{aligned} x_n \le x_{n+1} &\Leftrightarrow 3x_n + 4 \le 3x_{n+1} + 4 \\ &\Leftrightarrow \sqrt{3x_n+4} \le \sqrt{3x_{n+1}+4} \\ &\Leftrightarrow x_{n+1} \le x_{n+2}. \end{aligned} \tag{2}$$

Hence $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$.

(c) Prove that $\lim\limits_{n\to\infty} x_n = 4$. [3]

Solution:
From parts (a),and (b) we have $1 < x_n \le x_{n+1} < 4$, for all $n \in \mathbb{N}$. Since $\{x_n\}$ is increasing bounded sequence, then by MCT $\{x_n\}$ is convergent. Also, since $1 < x_n < 4$, then $1 \le \lim\limits_{n\to\infty} x_n \le 4$. Now, let $\lim\limits_{n\to\infty} x_n = x$, then $\lim\limits_{n\to\infty} x_{n+1} = x$ also. Since $x_{n+1} = \sqrt{3x_n+4}$, then $x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} (\sqrt{3x_n+4}) = \sqrt{\lim\limits_{n\to\infty} 3x_n + 4}$. Hence $x = \sqrt{3x+4}$. Thus $x^2 = 3x+4$. Hence $x^2 - 3x - 4 = 0$. Hence $(x+1)(x-4) = 0$. Thus $x = -1$, or $x = 4$. But since $1 \le x \le 4$, then $x \ne -1$. Therefore $\lim\limits_{n\to\infty} x_n = 4$.

3. Put (**T**) if the statement is true and (**F**) if the statement is false.

   (a) Every bounded sequence is *Cauchy.* [3]

   (a) ______________

   Solution:
   F.
   For example $x_n = (-1)^n$, then $|x_n| = |(-1)^n| = 1$ and hence bounded. But $\{x_n\}$ is divergent and hence not Cauchy.

   (b) The sequence $\{(-1)^n + \cos n\}$ has a Cauchy subsequence. [3]

   (b) ______________

   Solution:
   T.
   Since $|(-1)^n + \cos n| \le |(-1)^n| + |\cos n| \le 1 + 1 = 2$, then $\{(-1)^n + \cos n\}$ has a convergent subsequence and hence a Cauchy subsequence.

   (c) A monotone sequence is Cauchy. [3]

   (c) ______________

   Solution:
   F.
   The sequence $\{n\}$ is a monotone sequence which is divergent. Hence can not be Cauchy.

   (d) Let $A = \{x \in \mathbb{Q} : x < 6\}$. Then A is open. [3]

   (d) ______________

   Solution:
   F. Since any open interval contains rational and irrational numbers, then for $x \in A$ we have for all $\delta > 0$, $(x-\delta, x+\delta) \not\subseteq A$.
   Hence A is not open.

   (e) Let $A = \left\{\dfrac{2n}{n+1} | n \in \mathbb{N}\right\}$. Then $A' = \{2\}$. [3]

   (e) ______________

   Solution:
   T.
   Since $\lim\limits_{n\to\infty} \dfrac{2n}{n+1} = 2$, then $A' = \{2\}$.

   (f) Let $a \in \mathbb{R}$. Then $\{a\}$ is a closed set. [3]

   (f) ______________

   Solution:
   T. Since $\{a\}^c = (-\infty, a) \cup (a, \infty)$, which is open (union of two open intervals). Then $\{a\}$ is a closed set.

(g) Every sequence has a monotone subsequence that is convergent. [3]

(g) ____________

Solution:
F.
The sequence $\{n\}$ has no convergent subsequence.

(h) $\mathbb{Q} \cap \mathbb{Q}^c$ is a closed set. [3]

(h) ____________

Solution:
T.
Since $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$ which is closed and open at the same time.

(i) Let $a \in \mathbb{R}$. Then $a \in \mathbb{Q}^{'} \cap (\mathbb{Q}^c)^{'}$. [3]

(i) ____________

Solution:
T.
Since $\mathbb{Q}^{'} = \mathbb{R}$ and $(\mathbb{Q}^c)^{'} = \mathbb{R}$, then $a \in \mathbb{Q}^{'} \cap (\mathbb{Q}^c)^{'} = \mathbb{R}$.

(j) Let $A = \left\{ \dfrac{n-1}{n} \mid n \in \mathbb{N} \right\} \cup \{1\}$. Then A is closed. [3]

(j) ____________

Solution:
T.Since $\lim\limits_{n\to\infty} \dfrac{n-1}{n} = 1$. Then $A' = \{1\} \subset A$, then A is closed.

4. (a) Let $\{x_n\}$ be a bounded above increasing sequence and $x = \sup\{x_n : n \in \mathbb{N}\}$. Prove that $\lim\limits_{n\to\infty} x_n = x$. [4]

Solution:

Let $\epsilon > 0$ be given. Since $x - \epsilon$ is not an upper bound of $\{x_n : n \in \mathbb{N}\}$, then there exist $N \in \mathbb{N}$ such that $x - \epsilon < x_N$. Now, if $n > N$, since $\{x_n\}$ is increasing sequence, then $x_N \leq x_n$. If $n > N \Rightarrow x - \epsilon < x_N \leq x_n \leq x < x + \epsilon$. Hence, if $n > N \Rightarrow x - \epsilon < x_n < x + \epsilon$. Thus, if $n > N \Rightarrow |x_n - x| < \epsilon$. Therefore $\lim\limits_{n\to\infty} x_n = x$.

(b) Let $\{y_n\}$ be a bounded below decreasing sequence and $y = \inf\{y_n : n \in \mathbb{N}\}$. Prove that $\lim\limits_{n\to\infty} y_n = y$. [4]

Solution:

Let $\epsilon > 0$ be given. Since $y + \epsilon$ is not a lower bound of $\{y_n : n \in \mathbb{N}\}$, then there exist $N \in \mathbb{N}$ such that $y_N < y + \epsilon$. Now, if $n > N$, since $\{y_n\}$ is decreasing sequence, then $y_n \leq y_N$. If $n > N \Rightarrow y - \epsilon < y \leq y_n \leq y_N < y + \epsilon$. Hence, if $n > N \Rightarrow y - \epsilon < y_n < y + \epsilon$. Thus, if $n > N \Rightarrow |y_n - y| < \epsilon$. Therefore $\lim\limits_{n\to\infty} y_n = y$.

(c) Is the sequence $\left\{1 - \dfrac{1}{n}\right\}$ increasing or decreasing? Prove your answer. [3]

Solution:

$$\begin{aligned}
\text{Since } &\Leftrightarrow n < n+1 \\
&\Leftrightarrow \frac{1}{n} > \frac{1}{n+1} \\
&\Leftrightarrow -\frac{1}{n} < -\frac{1}{n+1} \\
&\Leftrightarrow 1 - \frac{1}{n} < 1 - \frac{1}{n+1}
\end{aligned}$$

Thus the sequence is increasing.

5. (a) Let A, B be two nonempty subsets of $\mathbb{R}$. []

i. If A is closed. Prove that $A' \subseteq A$. [4]

Solution:
Let $x \in A'$. Thus x is a limit point of A. We want to show that $x \in A$. Assume that $x \in A^c$. Since A is closed, then A^c is open and $x \in A^c$, then there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset A^c$. Hence $(x-\delta, x+\delta) \cap A = \emptyset$. Contradiction to the fact that x is a limit point of A. Hence $x \in A$.

ii. If A and B are open. Prove that $A \cup B$ is open. [4]

Solution:
Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Then there are $\delta_1, \delta_2 > 0$ such that $(x-\delta_1, x+\delta_1) \subseteq A$ and $(x-\delta_2, x+\delta_2) \subseteq B$. Let $\delta = \delta_1 > 0$. Now $(x-\delta, x+\delta) \subseteq A \subseteq A \cup B$. Hence $A \cup B$ is open.

(b) Give an example of two nonempty subsets A, B of $\mathbb{R}$ such that $A \cap B = \emptyset$, $A' = B'$, and $A^\circ = B^\circ$. [3]

Solution:
Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $\mathbb{Q}' = \mathbb{R} = (\mathbb{Q}^c)'$ and $\mathbb{Q}^\circ = \emptyset = (\mathbb{Q}^c)^\circ$.

6. Let $A = \{x \in \mathbb{Q} \mid 1 < x < 3\}, B = (4, 6]$, and $C = \left\{\pi + \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$.

   (a) Find A', B' and C'. Prove your answer. [4]

   Solution:
   For $x \in [1, 3]$ and for any $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains infinitely many rational numbers and hence $(x - \delta, x + \delta) \cap A - \{x\} \neq \emptyset$. Hence x is a limit point for A. Thus $A' = [1, 3]$.
   For $x \in [4, 6]$ and for any $\delta > 0$, the interval $(x - \delta, x + \delta)$ contains infinitely many numbers of B and hence $(x - \delta, x + \delta) \cap B - \{x\} \neq \emptyset$. Hence x is a limit point for B. Thus $B' = [4, 6]$.
   Since $\lim\limits_{n\to\infty} \left[\pi + \dfrac{1}{n}\right] = \pi$. Then $C' = \{\pi\}$.

   (b) Find A°, B° and C°. Prove your answer. [4]

   Solution:
   Since A and C contain no intervals, then $A^\circ = \emptyset = C^\circ$.
   For $x \in (4, 6)$, let $\delta = \min\left\{\dfrac{x-4}{2}, \dfrac{6-x}{2}\right\} > 0$. Then $(x - \delta, x + \delta) \subseteq B$. Hence x is an intrior point. Thus $B^\circ = (4, 6)$.

   (c) Prove that there exist a sequence $\{x_n\} \subset A$ such that $\lim\limits_{n\to\infty} x_n = \sqrt{2}$. [3]

   Solution:
   For each $n \in \mathbb{N}$, let $\alpha_n = \max\{1, \sqrt{2} - \frac{1}{n}\}$, and $\beta_n = \min\{3, \sqrt{2} + \frac{1}{n}\}$. Note that $1 \le \alpha_n < \beta_n \le 3$ and $\sqrt{2} - \frac{1}{n} \le \alpha_n < \beta_n \le \sqrt{2} + \frac{1}{n}$. Since between any real numbers there is a rational number, then the exists $x_n \in (\alpha_n, \beta_n)$ such that $x_n \in \mathbb{Q}$. Since $(\alpha_n, \beta_n) \subset (1, 3)$, and $x_n \in \mathbb{Q}$, then $x_n \in A$. Also since $(\alpha_n, \beta_n) \subset (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n})$, then $\sqrt{2} - \frac{1}{n} < x_n < \sqrt{2} + \frac{1}{n}$. Using Squeeze Theorem we have

   $$\sqrt{2} = \lim_{n\to\infty} [\sqrt{2} - \frac{1}{n}] \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} [\sqrt{2} + \frac{1}{n}] = \sqrt{2}.$$

   Thus $\lim\limits_{n\to\infty} x_n = \sqrt{2}$.

7. (a) Let $a \in \mathbb{R}$. Prove that $(-\infty, a)$ is open set. [4]

Solution:
For $x \in (-\infty, a)$ let $\delta = a - x > 0$. Since $-\infty < 2x - a = x - (a-x) = x - \delta < x + \delta = x - (a - x) = a$, then $(x-\delta, x+\delta) \subseteq (-\infty, a)$. Hence $(-\infty, a)$ is open set.

(b) Let $a, b \in \mathbb{R}$ with $a < b$. Prove that $\{a, b\}$ is closed set. [4]

Solution:
Since $\{a, b\}^c = (-\infty, a) \cup (a, b) \cup (b, \infty)$ which is open (union of open sets is open set), hence $\{a, b\}$ is closed set.

(c) Let $a \in \mathbb{R}$. Prove that there exist a sequence $\{x_n\} \subset (-\infty, a)$ such that $\lim\limits_{n\to\infty} x_n = a$. [3]

Solution:
For each $n \in \mathbb{N}$, since $a - \frac{1}{n} < a$, there exists $a - \frac{1}{n} < x_n < a < a + \frac{1}{n}$. Note that $x_n < a$ and $a - \frac{1}{n} \le x_n < a + \frac{1}{n}$. Hence $x_n \in (-\infty, a)$. Using Squeeze Theorem we have

$$a = \lim_{n\to\infty} [a - \frac{1}{n}] \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} [a + \frac{1}{n}] = a.$$

Thus $\lim\limits_{n\to\infty} x_n = a$.