بسم الله الرحمن الرحيم

- الرجاء كتابة اسمُك و رقمك الجامعي في المكان المخصص أسفل هذه الصفحة.
- تأكد من حصولك على جميع الأسئلة.
- الرجاء أغلق الجوال و عدم أستخدامه خلال الاختبار.
- يحتوي هذا الإختبار على **سبعة** أسئلة والمطلوب الأجابة على **ستة** منها، علماً بأن السؤالين **الأول و الثالث** أجباريان.
- الزمن المحدد لهذا الإختبار 120 دقيقة.
- بالتوفيق إن شاء الله.

Question:	1	2	3	4	5	6	7	Total
Points:	25	11	30	11	11	11	11	110
Score:								

Name : ____________________

Student's I.N.: ____________________

1. Let α, β, $x \in \mathbb{R}$, $\{x_n\}$ be a sequence of real numbers, and let $A \subseteq \mathbb{R}$ be bounded set. State the definition of the following:

   (a) α is *the supremum* of A. [5]

   Solution:
   α is the supremum of A if it satisfies the conditions:

   (1) α is an upper bound of A (i.e. $a \leq \alpha$ for all $a \in A$.), and

   (2) If v is any upper bound of A then $\alpha \leq v$.

   (b) β is *the infimum* of A. [5]

   Solution:
   β is *the infimum* of A if it satisfies the conditions:

   (1) β is a lower bound of A (i.e. $\beta \leq a$ for all $a \in A$.), and

   (2) If t is any lower bound of A, then $t \leq \beta$.

   (c) The *completeness axiom* of $\mathbb{R}$. [5]

   Solution:
   Every nonempty subset of $\mathbb{R}$ that has an upper bound also has a supremum in $\mathbb{R}$.

   (d) The *Density of* $\mathbb{Q}$. [5]

   Solution:
   If $a, b \in \mathbb{R}$ with $a < b$, then there exist a rational number $r \in \mathbb{Q}$ such that $a < r < b$.

   (e) The sequence $\{x_n\}$ *converges* to x. [5]

   Solution:
   The sequence $\{x_n\}$ *converges* to $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a natural number $N = N(\epsilon) \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \epsilon$, and we write $\lim\limits_{n\to\infty} x_n = x$.

2. (a) Let $\{x_n\}$ be a sequence and $x, y \in \mathbb{R}$. If $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} x_n = y$. Prove that $y = x$ [3]

Solution:
Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} x_n = x$, then there exist $N_1 \in \mathbb{N}$ such that if $n > N_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} x_n = y$, then there exist $N_2 \in \mathbb{N} \ni$ if $n > N_2 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Now, Let $N = \max\{N_1, N_2\}$. If $n > N$, then $n > N_1, \Rightarrow |x_n - x| < \frac{\epsilon}{2}$ and if $n > N$, then $n > N_2, \Rightarrow |x_n - y| < \frac{\epsilon}{2}$. Then $|x - y| = |x - x_{N+1} + x_{N+1} - y| \leq |x - x_{N+1}| + |x_{N+1} - y| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $0 \leq |x - y| \leq \epsilon$. Thus $|x - y| = 0$. Therefore $x = y$.

(b) Let $x_n = \frac{n+1}{2n+1}$. Use the ***definition*** of the limit of a sequence to prove that $\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}$. [5]

Solution:
Discussion:
We start with $\epsilon > 0$ and want to find $N = N(\epsilon) \in \mathbb{N}$ such that if $n > n \Rightarrow \left|\frac{n+1}{2n+1} - \frac{1}{2}\right| < \epsilon$.

$$\begin{aligned}
\left|\frac{n+1}{2n+1} - \frac{1}{2}\right| &= \left|\frac{2(n+1) - (2n+1)}{2(2n+1)}\right| \\
&= \left|\frac{2n+2-2n-1}{4n+2}\right| \\
&= \left|\frac{1}{4n+2}\right| \\
&\leq \frac{1}{4n+1} \\
&= \frac{1}{4n} && \text{Note that: } 4n+1 \geq 4n \\
&\leq \frac{1}{4n} && \text{Note that: } 4n+1 \geq 4n \Leftrightarrow \frac{1}{4n+1} \leq \frac{1}{4n} \\
&= \frac{1}{4n}.
\end{aligned}$$

Now, let $\frac{1}{4n} < \epsilon \quad \Leftrightarrow n > \frac{1}{4\epsilon}$.

Now, since $\frac{1}{4\epsilon}$ may not by an natural number, we let $N = N(\epsilon) > \frac{1}{4\epsilon}$.

Proof:
Let $\epsilon > 0$ be given. Let $N \in \mathbb{N}$ such that $N > \frac{1}{4\epsilon}$.

$$\text{Now, if } n > N \Rightarrow \frac{1}{n} < \frac{1}{N} < 4\epsilon$$
$$\Rightarrow \frac{1}{4n} < \epsilon$$
$$\Rightarrow \left|\frac{n+1}{2n+1} - \frac{1}{2}\right| < \frac{1}{4n} < \epsilon$$
$$\text{Now, if } n > N \Rightarrow \left|\frac{n+1}{2n+1} - \frac{1}{2}\right| < \epsilon.$$
$$\text{Therefore } \lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2}.$$

(c) Let $\{x_n\}$ be a sequence of real numbers such that $\sin\left(\frac{1}{n}\right) \le \frac{x_n}{n} \le \frac{n+1}{n^2}$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n = 1$. [3]

Solution:

$$\text{We have } \sin\left(\frac{1}{n}\right) \le \frac{x_n}{n} \le \frac{n+1}{n^2} \qquad \text{Multiply all sides by } n$$
$$n\sin\left(\frac{1}{n}\right) \le x_n \le \frac{n+1}{n} \qquad \text{Take the limits}$$
$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} \frac{n+1}{n}$$
$$\text{Now, } 1 \le \lim_{n\to\infty} x_n \le 1 \qquad \text{Using Squeeze Theorem.}$$
$$\text{Therefore } \lim_{n\to\infty} x_n = 1.$$

3. Put (**T**) if the statement is true and (**F**) if the statement is false.

(a) Every sequence is *bounded.* [3]

(a) ____________

Solution:
F. The sequence $\{2^n\}_{n=1}^{\infty}$ is unbounded sequence.

(b) Let A be a finite subset of $\mathbb{R}$. Then $\inf A = \min A$. [3]

(b) ____________

Solution:
T.Note that $\min A \in A$. Since $\min A \leq a, \forall a \in A$, then $\min A$ is a lower bound of A. Let b be any lower bound of A. Since $\min A \in A$ and b is a lower bound of A, then $b \leq \min A$. Hence $\inf A = \min A$.

(c) Let A, B be bounded two nonempty subsets of $\mathbb{R}$. If $A \subseteq B$, then $\inf A \geq \inf B$. [3]

(c) ____________

Solution:
T. Since $A \subseteq B$, then $\inf B \leq a, \forall a \in A$, then $\inf B$ is a lower bound for A. Hence $\inf B \leq \inf A$.

(d) Let $A = \{x \in \mathbb{Q} : x^2 < 16\}$. Then $\sup A \in \mathbb{Z}$. [3]

(d) ____________

Solution:
T. Note that $0 \in A$. Since $x^2 < 16, \forall x \in A$, then $-4 < x < 4$. Hence 4 is an upper bound for A. Let α be an upper bound for A. Suppose that $\alpha < 4$. Then by the density theorem for rational numbers there is $y \in \mathbb{Q}$ such that $\alpha < y < 4$. Now we have $0 \leq \alpha < y < 4$. Therefore $y^2 < 16$ and hence $y \in A$. But we have $\alpha < y$ and α is an upper bound for A. A contradiction. Thus $4 \leq \alpha$. Therefore $\sup A = 4 \in \mathbb{Z}$.

(e) If $a \leq b$, then $\dfrac{1}{a} \leq \dfrac{1}{b}$. [3]

(e) ____________

Solution:
F.
$-4 < -2$ but $\dfrac{-1}{4} > \dfrac{-1}{2}$.

(f) Let $a \in \mathbb{R}$. If $|a| < 1$, then the sequence $\{a^n\}_{n=1}^{\infty}$ is bounded. [3]

(f) ____________

Solution:
T. Since $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$. Hence $\{a^n\}_{n=1}^{\infty}$ is convergent and hence is bounded.

(g) Let $\{a_n\}, \{b_n\} \subset \mathbb{R}$ be two sequences of real numbers. If $a_n < b_n \forall n \in \mathbb{N}$, then [3]

$$\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n.$$

(g) ____________

Solution:
F. Let $a_n = 1 - \frac{1}{n}, b_n = 1 + \frac{1}{n}$. Then $a_n = 1 - \frac{1}{n} < 1 + \frac{1}{n} = b_n$. But $\lim_{n\to\infty} a_n = 1 = \lim_{n\to\infty} b_n$.

(h) If $a > 0$, there exist $n \in \mathbb{N}$ such that $\frac{1}{n} < a$. [3]

(h) ____________

Solution:
T. Since $\frac{1}{a} > 0$, then by Archimedean Property there is $n \in \mathbb{N}$ such that $\frac{1}{a} < n$. Hence $\frac{1}{n} < a$.

(i) Let A be nonempty subset of $\mathbb{R}$, and $c \in \mathbb{R}$. If $c < 0$, then $\sup(cA) = c \sup A$. [3]

(i) ____________

Solution:
F. For example, if $A = \{1, 3, -1\}$ and $c = -2 < 0$, then $\sup A = 3$, but $-2A = \{-2, -6, 2\}$ and hence $\sup(-2A) = 2 \neq -6 = -2 \sup A$.

(j) Let $A = \left\{ \frac{n-1}{n} \mid n \in \mathbb{N} \right\}$. Then $\sup A = 1$. [3]

(j) ____________

Solution:
T.Since $\frac{n-1}{n} = 1 - \frac{1}{n} < 1, \forall n \in \mathbb{N}$. Then 1 is an upper bound for A. Let $\epsilon > 0$ be given. There is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Hence $-\epsilon < -\frac{1}{n_0}$. Thus $1 - \epsilon < 1 - \frac{1}{n_0} = \frac{n_0 - 1}{n_0} \in A$. Thus $\sup A = 1$.

4. (a) If $a > 0$. Prove that then there is $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$ [3]

Solution:
Since $a \in \mathbb{R}$, then by Archimedean Property there is $n_1 \in \mathbb{N}$ such that $a < n_1$. Also since $\frac{1}{a} \in \mathbb{R}$, then by Archimedean Property there is $n_2 \in \mathbb{N}$ such that $\frac{1}{a} < n_2$. Hence $a > \frac{1}{n_2}$. Let $n = \max\{n_1, n_2\}$. Now, $a < n_1 < n$. Also $\frac{1}{n} < \frac{1}{n_2} < a$. Then $\frac{1}{n} < a < n$.

(b) Let $x \in \mathbb{R}$. Prove that there exist $n \in \mathbb{Z}$ such that $n \leq x < n + 1$. [3]

Solution:
By Archimedean Property, there exist $m \in \mathbb{N}$ such that $|x| < m$. Hence $-m < x < m$. The set $A_x = \{-m, -m+1, \ldots, 0, 1, \ldots, m-1, m\}$ is a finite set. The set $B_x = \{k : k \in A_x \text{ and } k \leq x\} \subset A_x$ is bounded above by x. Let $n = \sup B_x \in B_x$. Then $n \leq x$ and $n + 1 \notin B_x$. Hence $n \leq x < n + 1$.

(c) Let $a, b \in \mathbb{R}$ such that $a < b$ and $b - a > 1$. Prove that there is $m \in \mathbb{Z}$ such that $a < m < b$. [5]

Solution:
Since $b - a > 1$, then $a + 1 < b$. By part (b) there is $m \in \mathbb{Z}$ such that $m \leq a + 1 < m + 1$. Hence $a + 1 < m + 1$ and $a < m$. Now, $m \leq a + 1 < b$. Thus $m < b$. Therefore $a < m < b$.

5. Let A, B be two nonempty bounded subsets of $\mathbb{R}$.

   (a) Prove that $\inf(A+B) = \inf A + \inf B$. [4]

   Solution:
   Since $\inf A \leq a$ for all $a \in A$ and $\inf B \leq b$ for all $b \in B$, then $\inf A + \inf B \leq a + b$ for all $a \in A$ and $b \in B$. Hence $\inf A + \inf B$ is a lower bound for $A + B$. Let u be any lower bound of $A + B$. Hence $u \leq a + b$ for all $a \in A$ and $b \in B$. Thus for a fix $b \in B$, we have $u \leq a + b$ for all $a \in A$. Hence $u - b \leq a$ for all $a \in A$. Therefore $u - b$ is a lower bound for A. Hence $u - b \leq \inf A$ and thus $u - \inf A \leq b$ for all $b \in B$. Thus $u - \inf A$ is a lower bound for B. Hence $u - \inf A \leq \inf B$. Thus $u \leq \inf A + \inf B$. Therefore $\inf(A + B) = \inf A + \inf B$.

   (b) Prove that $\inf(-B) = -\sup B$. [4]

   Solution:
   Let $-B = \{-b : b \in B\}$. Since B is bounded below there is $m \in \mathbb{R}$ such that $m \leq b$ for all $b \in B$. Hence $-b \leq -m$ for all $b \in B$. Thus the set $-B$ is bounded above. Then by Completeness axiom $\sup(-B)$ exist and is a real number. Let $\alpha = \sup(-B)$ Now, $-b \leq \alpha$ for all $b \in B$. Hence $-\alpha \leq b$ for all $b \in B$. Thus $-\alpha$ is a lower bound for B. Let β be a lower bound for B. hence $\beta \leq b$ for all $b \in B$. Thus $-b \leq -\beta$ for all $b \in B$. Thus $-\beta$ is an upper bound for $-B$, but $\alpha = \sup(-B)$ and hence $\alpha \leq -\beta$ and therefore $\beta \leq -\alpha$. Thus $\inf B = -\alpha \in \mathbb{R}$. Hence $\inf B = -\sup(-B)$.

   (c) Prove that $\inf(A - B) = \inf A - \sup B$. [3]

   Solution:
   Note that $A - B = A + (-B)$. By part **(a)** and **(b)**, we have

   $$\inf(A-B) = \inf(A+(-B)) = \inf A + \inf(-B) = \inf A + (-\sup B) = \inf A - \sup B.$$

6. Let $A = \left\{x \in \mathbb{Q} \mid x^2 < 7\right\}$.

   (a) Prove that $\sup A = \sqrt{7}$. [4]

   Solution:
   Note that $0 \in A$. Since $x^2 < 7 \Leftrightarrow -\sqrt{7} < x < \sqrt{7}$ hence $\sqrt{7}$ is an upper bound for A. Now, if u is an upper bound for A. Since $0 \in A$, then $0 \leq u$. Suppose that $0 \leq u < \sqrt{7}$ then by density of $\mathbb{Q}$ there is $x \in \mathbb{Q}$ such that $0 \leq u < x < \sqrt{7}$. Now, since $0 \leq x < \sqrt{7} \Leftrightarrow x^2 < 7$, hence $x \in A$. But $u < x$ and u is an upper bound for A. Contradiction. Hence $\sqrt{7} \leq u$ and therefore $\sup A = \sqrt{7}$.

   (b) Prove that $\inf A = -\sqrt{7}$. [4]

   Solution:
   Note that $0 \in A$. Since $x^2 < 7 \Leftrightarrow -\sqrt{7} < x < \sqrt{7}$ hence $-\sqrt{7}$ is a lower bound for A. Now, if v is a lower bound for A. Since $0 \in A$, then $v \leq 0$. Suppose that $-\sqrt{7} < v \leq 0$ then by density of $\mathbb{Q}$ there is $y \in \mathbb{Q}$ such that $-\sqrt{7} < y < v \leq 0$. Now, since $-\sqrt{7} < y \leq 0 \Leftrightarrow y^2 < 7$, hence $y \in A$. But $y < v$ and v is a lower bound for A. Contradiction. Hence $v \leq -\sqrt{7}$ and therefore $\inf A = -\sqrt{7}$.

   (c) Prove that there exist a sequence $\{x_n\} \subset A$ such that $\lim_{n\to\infty} x_n = \sqrt{7}$. [3]

   Solution:
   For each $n \in \mathbb{N}$, since $\sqrt{7} - \dfrac{1}{n}$ is not an upper bound of A, then there is $x_n \in A$ such that
   $$\sqrt{7} - \frac{1}{n} < x_n \leq \sqrt{7} < \sqrt{7} + \frac{1}{n}.$$
   Thus
   $$\sqrt{7} - \frac{1}{n} < x_n < \sqrt{7} + \frac{1}{n}, \forall n \in \mathbb{N}.$$
   Using Squeeze Theorem we have
   $$\sqrt{7} = \lim_{n\to\infty}[\sqrt{7} - \frac{1}{n}] \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty}[\sqrt{7} + \frac{1}{n}] = \sqrt{7}.$$
   Thus $\lim_{n\to\infty} x_n = \sqrt{7}$.

7. (a) Prove that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$. [4]

Solution:
Since $a = a - b + b$, then $|a| = |a - b + b| \leq |a - b| + |b|$, and hence $|a| - |b| \leq |a - b|$. Also, since $b = b - a + a$, then $|b| = |b - a + a| \leq |b - a| + |a|$, and hence $|b| - |a| \leq |b - a| = |a - b|$. Now, $|b| - |a| \leq |a - b| \Leftrightarrow -|a - b| \leq -|b| + |a|$ and hence $-|a - b| \leq |a| - |b|$. Also we have $|a| - |b| \leq |a - b|$. Thus $-|a - b| \leq |a| - |b| \leq |a| - |b| \leq |a - b|$. Therefore $||a| - |b|| \leq |a - b|$.

(b) Let $\{x_n\}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. Prove that $\lim\limits_{n\to\infty} |x_n| = |x|$. [4]

Solution:
We have that $||a| - |b|| \leq |a - b|,\ \forall a, b \in \mathbb{R}$. Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to\infty} x_n = x$, therefore there exists $N \in \mathbb{N} \ni$ if $n > N \Rightarrow |x_n - x| < \epsilon$. Now,if $n > N \Rightarrow ||x_n| - |x|| \leq |x_n - x| < \epsilon$. Thus $\lim\limits_{n\to\infty} |x_n| = |x|$.

(c) Let $\{a_n\}$ be a sequence such that $|a_n - 5| \leq \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$. Prove that $\lim\limits_{n\to\infty} a_n = 5$. [3]

Solution:
Let $\epsilon > 0$ be given. Since $\lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$, therefore there exists $N \in \mathbb{N} \ni$ if $n > N \Rightarrow \dfrac{1}{n^2} = \left|\dfrac{1}{n^2}\right| < \epsilon$. Now,if $n > N \Rightarrow |a_n - 5| \leq \dfrac{1}{n^2} = \left|\dfrac{1}{n^2}\right| < \epsilon$. Thus $\lim\limits_{n\to\infty} a_n = 5$.