MATH 204

Chapter 4

The Laplace Transform

4.1 Definition of the Laplace Transform

The Laplace Transform of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

provided that the integral converges.

Example. Evaluate the Laplace transform of the following functions :

1. $f(t) = a$

2. $f(t) = t$

3. $f(t) = e^{at}$

4. $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & 3 \leq t. \end{cases}$

Laplace Transforms of Some Basic Functions :

1. $\mathcal{L}\{a\} = \dfrac{a}{s}$

2. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, \ldots$

3. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

4. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$

5. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

6. $\mathcal{L}\{\sinh at\} = \dfrac{a}{s^2-a^2}$

7. $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Remark. The Laplace transform is linear :

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}.$$

Example. Evaluate the Laplace transform of the following functions :

1. $f(t) = t^2 + e^{-9t} - 3$

2. $f(t) = (2t-1)^3$

3. $f(t) = e^t \sinh t$

4. $f(t) = \cos^2 t$

5. $f(t) = 14\cos\left(t - \frac{\pi}{6}\right)$

(Exercises 4.1)

Q1. Evaluate the Laplace transform of the following functions :

1. $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & 1 \leq t. \end{cases}$
2. $f(t) = t^2 + 6t - 3$
3. $f(t) = (t+1)^3$
4. $f(t) = 1 + e^{4t}$
5. $f(t) = 4t^2 - 5\sin 3t$
6. $f(t) = e^{-t}\cosh t$
7. $f(t) = \sin 2t \cos 2t$
8. $f(t) = \sin(4t + 5)$

4.2 Inverse Laplace Transform

If $F(s)$ is the Laplace transform of $f(t)$, that is,

$$F(s) = \mathcal{L}\{f(t)\},$$

then $f(t)$ is called the inverse Laplace transform of $F(s)$, and we write

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Some Inverse Laplace Transforms :

1. $\mathcal{L}^{-1}\left\{\frac{a}{s}\right\} = a$
2. $\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1, 2, \ldots$
3. $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
4. $\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
5. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
6. $\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$
7. $\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Example. Evaluate the inverse Laplace transform of the following functions :

1. $F(s) = \dfrac{1}{s^4}$

2. $F(s) = \dfrac{1}{s+2}$

3. $F(s) = \dfrac{1}{s^2+9}$

Remark. The inverse Laplace transform is linear :

$$\mathcal{L}^{-1}\{c_1F(s) + c_2G(s)\} = c_1\mathcal{L}^{-1}\{F(s)\} + c_2\mathcal{L}^{-1}\{G(s)\}.$$

Example. Evaluate the inverse Laplace transform of the following functions :

1. $F(s) = \dfrac{-2s+6}{s^2+4}$

2. $F(s) = \dfrac{(s+2)^2}{s^3}$

3. $F(s) = \dfrac{1}{s^2-3s}$

4. $F(s) = \dfrac{s}{s^2 + 2s - 3}$

5. $F(s) = \dfrac{s}{(s-2)(s-3)(s-6)}$

6. $F(s) = \dfrac{s}{(s+2)(s^2+4)}$

(Exercises 4.2)

Q1. Evaluate the inverse Laplace transform of the following functions :

1. $F(s) = \dfrac{(s+1)^3}{s^4}$
2. $F(s) = \dfrac{5}{s^2 + 49}$
3. $F(s) = \dfrac{s+1}{s^2+2}$
4. $F(s) = \dfrac{1}{s^2 + s - 20}$
5. $F(s) = \dfrac{1}{s^3 + 5s}$

4.3 Properties of the Laplace Transform

Property 1.

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a}$$

Example. Evaluate the following Laplace transforms :

1. $\mathcal{L}\{e^{2t}t^3\}$

2. $\mathcal{L}\{e^{-2t}\cos 4t\}$

3. $\mathcal{L}\left\{t(e^t + e^{2t})^2\right\}$

Remark. From property 1, we have

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\,\mathcal{L}^{-1}\{F(s)\}.$$

Example. Evaluate the following inverse Laplace transforms :

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)^4}\right\}$

2. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-6s+10}\right\}$

3. $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4s+5}\right\}$

Property 2. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

For example,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \qquad \text{and} \qquad \mathcal{L}\{t^2 f(t)\} = \frac{d^2}{ds^2}F(s).$$

Example. Evaluate the following Laplace transforms :

1. $\mathcal{L}\{t \sin 3t\}$

2. $\mathcal{L}\{t^3 e^t\}$

Definition. The convolution of two functions f and g is defined by

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Property 3.

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Example. Evaluate the following Laplace transforms :

1. $\mathcal{L}\{1 * t^3\}$

2. $\mathcal{L}\left\{\int_0^t e^\tau \sin(t-\tau)\, d\tau\right\}$

(Exercises 4.3)

Q1. Evaluate the following Laplace transforms :

1. $\mathcal{L}\{e^{-2t}t^3\}$
2. $\mathcal{L}\{e^{2t}(t-1)^2\}$
3. $\mathcal{L}\{e^t \sin 3t\}$
4. $\mathcal{L}\{te^{-10t}\}$
5. $\mathcal{L}\{t\cos 2t\}$
6. $\mathcal{L}\{e^{2t} * \sin t\}$
7. $\mathcal{L}\left\{\int_0^t \sin\tau \cos(t-\tau)\, d\tau\right\}$

Q2. Evaluate the following inverse Laplace transforms :

1. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\}$
2. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s+5}\right\}$
3. $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{s^2+6s+34}\right\}$

4.4 Solving Initial Value Problems Using Laplace Transform

The Laplace transform of the nth derivative of $f(t)$ is given by

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0).$$

For example,

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \qquad \text{and} \qquad \mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

The Laplace transform can be used to solve initial value problems.

Example. Solve the following initial value problems :

1. $\dfrac{dy}{dt} - y = 1, \qquad y(0) = 0$

2. $y' + 3y = 13\sin 2t, \qquad y(0) = 6$

3. $y'' + 9y = e^{-t}, \qquad y(0) = 0, \quad y'(0) = 0$

(Exercises 4.4)

Q1. Solve the following initial value problems :

1. $y' + 6y = e^{4t}, \qquad y(0) = 2$
2. $y' - y = 2\cos 5t, \qquad y(0) = 0$
3. $y'' + 5y' + 4y = 0, \qquad y(0) = 1, \quad y'(0) = 0$

(Answers 4.1)

A1.

1. $F(s) = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2}$
2. $F(s) = \dfrac{2}{s^3} + \dfrac{6}{s^2} - \dfrac{3}{s}$
3. $F(s) = \dfrac{6}{s^4} + \dfrac{6}{s^3} + \dfrac{3}{s^2} + \dfrac{1}{s}$
4. $F(s) = \dfrac{1}{s} + \dfrac{1}{s-4}$
5. $F(s) = \dfrac{8}{s^3} - \dfrac{15}{s^2+9}$
6. $F(s) = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{1}{s+2}\right]$
7. $F(s) = \dfrac{2}{s^2+16}$
8. $F(s) = \dfrac{4\cos 5 + s\sin 5}{s^2+16}$

(Answers 4.2)

A1.

1. $f(t) = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3$
2. $f(t) = \frac{5}{7}\sin 7t$
3. $f(t) = \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t$
4. $f(t) = \frac{1}{9}e^{4t} - \frac{1}{9}e^{-5t}$
5. $f(t) = \frac{1}{5} - \frac{1}{5}\cos\sqrt{5}t$

(Answers 4.3)

A1.

1. $F(s) = \dfrac{6}{(s+2)^4}$
2. $F(s) = \dfrac{2}{(s-2)^3} - \dfrac{2}{(s-2)^2} + \dfrac{1}{(s-2)}$
3. $F(s) = \dfrac{3}{(s-1)^2+9}$
4. $F(s) = \dfrac{1}{(s+10)^2}$
5. $F(s) = \dfrac{s^2-4}{(s^2+4)^2}$
6. $F(s) = \dfrac{1}{(s-2)(s^2+1)}$
7. $F(s) = \dfrac{s}{(s^2+1)^2}$

A2.

1. $f(t) = \frac{1}{2}t^2e^{-2t}$
2. $f(t) = \frac{1}{2}e^{-t}\sin 2t$
3. $f(t) = 2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$

(Answers 4.4)

A1.

1. $y = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}$
2. $y = \frac{1}{13}e^{t} - \frac{1}{13}\cos 5t + \frac{5}{13}\sin 5t$
3. $y = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$