Cauchy Sequence

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5.1 Cauchy Sequence

Definition 5.1: A sequence $\{x_n\}$ of real numbers is said to be ***Cauchy sequence*** if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N \Rightarrow |x_n - x_m| < \epsilon$.

Note 5.1: A sequence is Cauchy if the terms eventually get arbitrarily close to each other.

Example 5.1: The sequence $\{\frac{1}{n}\}$ is Cauchy. To see this let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Now, if $n, m > N \Rightarrow \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Example 5.2: The sequence $\{\frac{n}{n+1}\}$ is Cauchy. To see this let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Now, if $n, m > N \Rightarrow \left|\frac{n}{n+1} - \frac{m}{m+1}\right| = \left|\frac{(m+1)n - m(n+1)}{(n+1)(m+1)}\right| \leq \left|\frac{n-m}{nm}\right| < \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Lemma 5.1: Let sequence $\{x_n\}$ be a Cauchy sequence of real numbers. Then $\{x_n\}$ is bounded.

Proof: Since $\{x_n\}$ is a Cauchy sequence, then there exists $N \in \mathbb{N}$ such that if $n, m > N \Rightarrow |x_n - x_m| < 3$.

$$\begin{aligned}
&\text{if } n, m > N \Rightarrow |x_n - x_m| < 3 \\
&\text{let } m = N+1, \text{ if } n > N \Rightarrow |x_n - x_{N+1}| < 3 \qquad \text{Note: } |x_n| - |x_{N+1}| \leq |x_n - x_{N+1}| \\
&\Rightarrow |x_n| - |x_{N+1}| \leq |x_n - x_{N+1}| < 3 \\
&\text{if } n > N \Rightarrow |x_n| < 3 + |x_{N+1}|. \\
&\text{Let } M = \max\{|x_1|, |x_2|, \cdots |x_N|, |x_{N+1}| + 3\} \\
&\text{Now, if } n > N \Rightarrow |x_n| < 3 + |x_{N+1}| \leq M \\
&\text{Now, if } n \leq N \Rightarrow |x_n| < \max\{|x_1|, |x_2|, \cdots |x_N|\} \leq M \\
&\text{Thus } \forall\ n \in \mathbb{N},\ |x_n| \leq M.
\end{aligned}$$

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Theorem 5.1: *[Cauchy Convergence Criterion]*

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Proof: Let $\{x_n\}$ be a sequence of real numbers.

($\Rightarrow$) Suppose that $\lim_{n\to\infty} x_n = x \in \mathbb{R}$. We want to show that $\{x_n\}$ is Cauchy sequence. Let $\epsilon > 0$ be given. Since $\lim_{x\to\infty} x_n = x$ then there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Also, if $m > N \Rightarrow |x_m - x| < \frac{\epsilon}{2}$. Now, if $n, m > N \Rightarrow |x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $\{x_n\}$ is a Cauchy sequence.

($\Leftarrow$) Suppose that $\{x_n\}$ is a Cauchy sequence.We want to show that $\{x_n\}$ is convergent. Let $\epsilon > 0$ be given. Since $\{x_n\}$ is a Cauchy sequence, then it is bounded. Hence $\{x_n\}$ has a converge subsequence $\{x_{n_k}\}$. Suppose $\lim\limits_{k\to\infty} x_{n_k} = x \in \mathbb{R}$. There exist $N_1, N_2 \in \mathbb{N}$ such that if $n, m > N_1 \Rightarrow |x_n - x_m| < \dfrac{\epsilon}{2}$ and, if $k > N_2 \Rightarrow |x_{n_k} - x| < \dfrac{\epsilon}{2}$. Now, fix $k > N_2$ such that $n_k > N_1$ and, if $n > N_1 \Rightarrow |x_n - x_{n_k}| < \dfrac{\epsilon}{2}$ and $|x_{n_k} - x| < \dfrac{\epsilon}{2}$. Now, if $n > N_1 \Rightarrow |x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$. Thus $\{x_n\}$ converges .

Example 5.3: Prove that any sequence of real numbers $\{x_n\}$ which satisfies $|x_n - x_{n+1}| = \dfrac{1}{5^n}, \quad \forall\ n \in \mathbb{N}$ is convergent.

Solution:

$$\begin{aligned}
\text{If } m > n \Rightarrow |x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} + x_{n+2} + \ldots + x_{m-1} - x_m| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} + x_{n+2}| + \ldots + |x_{m-1} - x_m| \\
&= \frac{1}{5^n} + \frac{1}{5^{n+1}} + \ldots + \frac{1}{5^{m-1}} \\
&= \frac{1}{5^{n-1}}\left(\frac{1}{5} + \frac{1}{5^2} + \ldots + \frac{1}{5^{m-n}}\right) \\
&= \frac{1}{5^{n-1}} \sum_{k=1}^{m-n} \frac{1}{5^k} \\
&= \frac{1}{5^{n-1}}\left(1 - \frac{1}{5^{m-n}}\right) && \text{Note that: } \left(1 - \frac{1}{5^{m-n}}\right) < 1 \\
&< \frac{1}{5^{n-1}}.
\end{aligned}$$

Let $\varepsilon > 0$ be given, choose $N \in \mathbb{N}$ such that $\dfrac{1}{5^{n-1}} < \epsilon$.

Now, if $n, m > N \Rightarrow |x_n - x_m| < \dfrac{1}{5^{n-1}} < \epsilon$.

Thus $\{x_n\}$ is convergent.