MATH 251
Assignment 2: Principle of Mathematical Induction

The due date for this assignment is ....

1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

   (a) What is the statement $P(1)$?

   (b) Show that $P(1)$ is true, completing the basis step of the proof.

   (c) What is the inductive hypothesis?

   (d) What do you need to prove in the inductive step?

   (e) Complete the inductive step, identifying where you use the inductive hypothesis.

   (f) Explain why these steps show that this formula is true whenever n is a positive integer.

2. Use Principle of Mathematical Induction to prove that:

   (1) $1^2+3^2+5^2+...+(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$, whenever n is a non-negative integer.

   (2) $2+4+6+...+2n = n\,(n+1)$, whenever n is a positive integer.

   (3) $1+4+7+...+(3n-2) = \frac{1}{2}\left[n\,(3n-1)\right]$ whenever n is a natural number.

   (4) $2+5+8+...+(3n-1) = \frac{1}{2}\left[n\,(3n+1)\right]$ whenever n is a natural number.

   (5) $1^2+2^2+...+n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$ whenever n is a positive integer.

   (6) $1^3+2^3+...+n^3 = \dfrac{n^2\,(n+1)^2}{4} = \left[\dfrac{n\,(n+1)}{2}\right]^2$ whenever n is an integer.

   (7) $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+...+\dfrac{1}{2^n} = 1-\dfrac{1}{2^n}$ whenever n is a natural number.

   (8) $1\cdot 1! + 2\cdot 2! + ... + n\cdot n! = (n+1)! - 1$, whenever n is a positive integer.

   (9) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+...+\dfrac{1}{n\,(n+1)} = \dfrac{n}{n+1}$ whenever n is a positive integer.

   (11) $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = 3(5^{n+1}-1)/4$ whenever n is a nonnegative integer.

(12) $a + ar + ar^2 + ... + ar^n = \dfrac{ar^{n+1} - a}{r - 1}$ whenever n is a nonnegative integer.

3. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.

   (a) What is the statement $P(2)$?

   (b) Show that $P(2)$ is true, completing the basis step of the proof.

   (c) What is the inductive hypothesis?

   (d) What do you need to prove in the inductive step?

   (e) Complete the inductive step.

   (f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

4. Use Principle of Mathematical Induction to prove the following inequalities:

   (1) $3^n < n!$ if n is an integer greater than 6.

   (2) $2^n > n^2$ if n is an integer greater than 4.

   (3) $n < 5^n$ whenever n is a positive integer.

5. Use Mathematical Induction to prove divisibility facts.

   (a) 2 divides $n^2 + n$ whenever n is a positive integer

   (b) 3 divides $n^3 + 2n$ whenever n is a positive integer.

   (c) 5 divides $n^5 - n$ whenever n is a nonnegative integer.

   (d) 24 divides $5^{2n} - 1$ whenever n is an integer.

   (e) 3 divides $2^{2n-1} + 1$ whenever n is a positive integer.