Exercises

"For example," is not proof.

Jewish Proverb

1. Give two reasons why the set of odd integers under addition is not a group.
2. Referring to Example 13, verify the assertion that subtraction is not associative.
3. Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.
4. Show that the group $GL(2, \mathbf{R})$ of Example 9 is non-Abelian by exhibiting a pair of matrices A and B in $GL(2, \mathbf{R})$ such that $AB \neq BA$.
5. Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, Z_{11})$.
6. Give an example of group elements a and b with the property that $a^{-1}ba \neq b$.
7. Translate each of the following multiplicative expressions into its additive counterpart. Assume that the operation is commutative.
   a. a^2b^3
   b. $a^{-2}(b^{-1}c)^2$
   c. $(ab^2)^{-3}c^2 = e$
8. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?
9. (From the GRE Practice Exam) Let p and q be distinct primes. Suppose that H is a proper subgroup of the integers under addition that contains exactly three elements of the set $\{p, p + q, pq, p^q, q^p\}$. Determine which of the following are the three elements in H.
   a. pq, p^q, q^p
   b. $p + q, pq, p^q$
   c. $p, p + q, pq$
   d. p, p^q, q^p
   e. p, pq, p^q
10. List the members of $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.
11. Prove that the set of all 2×2 matrices with entries from $\mathbf{R}$ and determinant $+1$ is a group under matrix multiplication.
12. For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

13. An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out? (This really happened!)

14. Let G be a group with the following property: Whenever a, b, and c belong to G and $ab = ca$, then $b = c$. Prove that G is Abelian. ("Cross cancellation" implies commutativity.)

15. (Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups?

16. (Socks-Shoes Property) Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes. Find an example that shows that in a group, it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. Find distinct nonidentity elements a and b from a non-Abelian group such that $(ab)^{-1} = a^{-1}b^{-1}$.

17. Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

18. Prove that in a group, $(a^{-1})^{-1} = a$ for all a.

19. For any elements a and b from a group and any integer n, prove that $(a^{-1}ba)^n = a^{-1}b^n a$.

20. If $a_1, a_2, \ldots, a_n$ belong to a group, what is the inverse of $a_1 a_2 \cdots a_n$?

21. The integers 5 and 15 are among a collection of 12 integers that form a group under multiplication modulo 56. List all 12.

22. Give an example of a group with 105 elements. Give two examples of groups with 44 elements.

23. Prove that every group table is a *Latin square*†; that is, each element of the group appears exactly once in each row and each column. (This exercise is referred to in this chapter.)

24. Construct a Cayley table for $U(12)$.

25. Suppose the table below is a group table. Fill in the blank entries.

	e	a	b	c	d
e	e	—	—	—	—
a	—	b	—	—	e
b	—	c	d	e	—
c	—	d	—	a	b
d	—	—	—	—	—

†Latin squares are useful in designing statistical experiments. There is also a close connection between Latin squares and finite geometries.

26. Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$.
27. Let a, b, and c be elements of a group. Solve the equation $axb = c$ for x. Solve $a^{-1}xa = c$ for x.
28. Prove that the set of all rational numbers of the form 3^m6^n, where m and n are integers, is a group under multiplication.
29. Let G be a finite group. Show that the number of elements x of G such that $x^3 = e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.
30. Give an example of a group with elements a, b, c, d, and x such that $axb = cxd$ but $ab \neq cd$. (Hence "middle cancellation" is not valid in groups.)
31. Let R be any rotation in some dihedral group and F any reflection in the same group. Prove that $RFR = F$.
32. Let R be any rotation in some dihedral group and F, any reflection in the same group. Prove that $FRF = R^{-1}$ for all integers k.
33. Suppose that G is a group with the property that for every choice of elements in G, $axb = cxd$ implies $ab = cd$. Prove that G is Abelian. ("Middle cancellation" implies commutativity.)
34. In the dihedral group D_n, let $R = R_{360/n}$ and let F be any reflection. Write each of the following products in the form R^i or R^iF, where $0 \leq i < n$.
    a. In D_4, $FR^{-2}FR^5$
    b. In D_5, $R^{-3}FR^4FR^{-2}$
    c. In D_6, $FR^5FR^{-2}F$
35. Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian. (This exercise is referred to in Chapter 26.)
36. Prove that the set of all 3×3 matrices with real entries of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is a group. (Multiplication is defined by

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & b'+ac'+b \\ 0 & 1 & c'+c \\ 0 & 0 & 1 \end{bmatrix}.$$

This group, sometimes called the *Heisenberg group* after the Nobel Prize–winning physicist Werner Heisenberg, is intimately related to the Heisenberg Uncertainty Principle of quantum physics.)