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Inferences for Burr Parameters Based on Censored Samples in Accelerated Life Tests

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Abstract: This paper studies the estimation problem in partially accelerated life tests in which test items are run simultaneously at normal conditions for a specified time, and the surviving items are then run at accelerated conditions until a predetermined censoring number of failures are observed. The lifetime distribution of the test items is assumed to follow Burr type III distribution. The maximum likelihood estimates are obtained for the distribution parameters and acceleration factor in type II censored samples. In addition, asymptotic variances and covariance matrix of the estimators are given. An iterative procedure is used to obtain the estimators numerically using *MathCAD*. Furthermore, confidence intervals of the estimators are presented. For illustrating the precision and variations of maximum likelihood estimators simulation results are included for different sample sizes.

Keywords: Maximum likelihood method; Type II censoring; Step stress accelerated life test; Accelerated factor, Burr type III distribution; Fisher information matrix.

1. Introduction

Under continuous quality and reliability improvement of products, it is more difficult to obtain failure information under normal condition. This makes lifetime test under these conditions very costive, take a long time and unusefull. For this reason, accelerated life test (ALT) is used to get information about the lifetime distribution of product or materials shortly and fastly. Accelerated life testing is achieved by subjecting the test units to conditions that are more severe than the normal ones, such as higher levels of temperature, voltage, pressure, vibration, cycling rate, load, *etc*. Research on ALT has commenced in the 1950's to develop a more effective testing technique. Chernoff[1] and Bessler *et al*.[2] introduced and studied the concept of accelerated life tests.

According to Nelson[3], the stress can be applied in various ways. One way to accelerate failure in step-stress, which increases the stress applied to test product in a specified discrete sequence. Step-stress partially accelerated life test (SS-PALT) is used to get quickly information for the lifetime of product with high reliability; specially, when the mathematical model related to test conditions of mean lifetime of the product is unknown and cannot be assumed. The step stress scheme applies stress to test units in the way that the stress will be changed at prespecified time. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it raised and held a specified time. Stress is repeatedly increased until test unit fails or censoring scheme is reached.

For an overview of SS-PALT, there is an amount of literature on designing SS-PALT. Goel[4] considered the estimation problem of the accelerated factor using both maximum likelihood and Bayesian methods for items having exponential and uniform distributions. DeGroot and Goel[5] estimated the parameters of the exponential distribution and acceleration factor in SS-PALT using Bayesian approach, with different loss functions. Also, Bhattacharyya and Soejoeti [6] estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method. Bai and Chung [7] estimated the scale parameter and acceleration factor for exponential distribution under type I censored sample using maximum likelihood method.

Ahmad and Islam[8] developed optimal accelerated life test designs for Burr Type XII distributions under periodic inspection and Type I censoring.. Attia *et al*.[9] considered the maximum likelihood method for estimating the acceleration factor and the parameters of Weibull distribution in SS-PALT under type I censoring. Abel-Ghaly *et al*.[10] used Bayesian approach for estimating the parameters of Weibull distribution parameters with known shape parameter. They studied the estimation problem in SS-PALT under both type I and type II censored data. Abdel-Ghani[11] considered the estimation problem of the parameters of Weibull distribution and the acceleration factor for both SS-PALT and constant-stress PALT. He applied maximum likelihood and Bayesian methods under type I and type II censored data. Abdel-Ghaly *et al*.[12] studied the estimation problem of the acceleration factor and the parameters of Weibull distribution in SS-PALT using maximum likelihood method in two types of data, namely type I and type II censoring.

Abdel-Ghaly *et al*.[13,14] studied both the estimation and optimal design problems for the Pareto distribution under SS-PALT with type I and type II censoring.

Abdel-Ghani[15] considered the estimation problem of log-logistic distribution parameters under SS-PALT.

Recently, Ismail[16] used maximum likelihood and Bayesian methods for estimating the acceleration factor and the parameters of Pareto distribution of the second kind. Ismail[17] studied the estimation and optimal design problems for the Gompertz distribution in SS-PALT with type I censored data. Abd-Elfattah *et al.*[18] studied the estimation problem of the acceleration factor and the parameters of Burr type XII distribution in SS-ALT using maximum likelihood method under type I censoring. For a concise review of step-stress ALT, readers may refer to Gouno *et al*[19]., Nelson , Li & Fard and Yang[19-22].

This article focuses on the maximum likelihood method for estimating the acceleration factor and the parameters of Burr type III distribution. This work was conducted for SS-PALT under type II censored sample. The performance of the obtained estimators is investigated in terms of relative absolute bias, mean square error and the standard error. Moreover, the confidence intervals of the estimators will be obtained.

The rest of this article can be organized as follows. In Section 2 the Burr type III distribution is introduced as a lifetime model and the test method is also described. Section 3 presents point and interval estimates of parameters and acceleration factor for the Burr type III under type II censoring using maximum likelihood method. In Section 4 the approximate asymptotic variances and covariances matrix are investigated. Section 5 explains the simulation studies for illustrating the theoretical results. Finally, conclusions are included in Section 6. Tables are displayed in the Appendix.

2. The Model Assumptions

This section introduces the assumed model for product life and also fully describes the test method.

Notation

ALT accelerated life testing.

OALT ordinary accelerated life testing.

PALT partially accelerated life testing.

SS-PALT step-stress partially accelerated life testing.

n total number of test items in a PALT.

T lifetime of an item at normal conditions.

Y total lifetime of an item in a SS-PALT.

y_i observed value of the total lifetime Y_i of item i , $i = 1,...,n$.

$y_{(r)}$ the time of the r^{th} failure at which the test is terminated.

$f(t)$ probability density function at time t at normal conditions.

$F(t)$ cumulative distribution function at time t at normal conditions.

$R(t)$ reliability function at time t at normal conditions.

β acceleration factor $(\beta > 1)$.

τ stress change time in a SS-PALT ($\tau < y_{(r)}$).

c, k the shape parameters of the Burr type III distribution.

δ_{1i}, δ_{2i} indicator functions in a SS-PALT: $\delta_{1i} = I(Y_i \le \tau)$, $\delta_{2i} = I(\tau < Y_i \le y_{(r)})$.

n_u, n_a number of items failed at normal and accelerated conditions respectively.

F Fisher information matrix.

$L(.)$ likelihood function.

$\ln L(.)$ the natural logarithm of likelihood function.

$y_{(1)} \le ... \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le ... \le y_{(r)}$ ordered failure times at the two conditions.

MSE mean square error.

ARBias absolute relative bias.

RE relative error.

After (1942), Burr[23] introduced a family of distributions which includes twelve types of cumulative distribution functions with a variety of density shapes. The Burr type III distribution with two parameters denoted by Burr (c, k) has density function of the form

$$f(x) = ck\, x^{-(c+1)} (1 + x^{-c})^{-(k+1)}, \quad x > 0, \quad c, k > 0, \quad (2.1)$$

where c and k are the shape parameters of the distribution.

The cumulative distribution function is

$$F(x) = (1 + x^{-c})^{-k} \quad , \quad x > 0 \tag{2.2}$$

2.1 The Test Method

In SS-PALT, all of the n units are tested first under normal condition, if the unit does not fail for a pre-specified time τ, then it runs at accelerated condition until failure. This means that if the item has not failed by some pre-specified time τ, the test is switched to the higher level of stress and it is continued until items fail. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the acceleration factor β. In this case the switching to the higher stress level will shorten the life of test item. Thus the total lifetime of a test item, denoted by Y, passes through two stages, which are the normal and accelerated conditions. Then the lifetime of the unit in SS-ALT is given as follows:

$$Y = \begin{cases} T & \text{if } T \le \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{2.3}$$

where, T is the lifetime of an item at use condition, τ is the stress change time and β is the acceleration factor which is the ratio of mean life at use condition to that at accelerated condition, usually $\beta > 1$. Assume that the lifetime of the test item follows Burr type III distribution with shape parameters c and k. Therefore, the probability density function of total lifetime Y of an item is given by:

$$f(y) = \begin{cases} 0 & \text{if } y \le 0 \\ f_1(y) & \text{if } 0 < y \le \tau \\ f_2(y) & \text{if } y > \tau \end{cases} \tag{2.4}$$

where, $f_1(y) = c\,k\,y^{-(c+1)}\left(1 + y^{-c}\right)^{-(k+1)} c, k > 0$, is the equivalent form to Equation (2.1), and,

$$f_2(y) = \beta c\, k \left[\tau + \beta(y - \tau)\right]^{-(c+1)} \left[1 + \left\{\tau + \beta(y - \tau)\right\}^{-c}\right]^{-(k+1)}, c, k > 0, \beta > 1,$$

is obtained by the transformation variable technique using equations (2.1) and (2.3).

3. Maximum Likelihood Estimation

The maximum likelihood is one of the most important and widely used methods in statistics. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the of the sample data. Furthermore, maximum likelihood estimators are consistent and asymptotically normally distributed.

In this section point and interval estimation for the parameters and acceleration factor of Burr type III distribution based on type II censoring are evaluated using maximum likelihood method.

3.1 Point Estimation

In type II censoring the test terminates when the censoring number of failure r is reached. The observed values of the total lifetime Y are $y_{(1)} \le \ldots \le y_{(n_u)} \le \tau \le y_{(n_u+1)} \le \ldots \le y_{(r)}$, where $r = n_u + n_a$ is the total number of failure in the test, n_u and n_a is the number of items failed at normal and accelerated conditions respectively. Let δ_{1i}, δ_{2i} be indictor functions, such that

$$\delta_{1i} = \begin{cases} 1 & y_{(i)} \le \tau \\ 0 & otherwise \end{cases} \qquad i = 1, 2, \ldots, n$$

and

$$\delta_{2i} = \begin{cases} 1 & \tau < y_{(i)} \le y_{(r)} \\ 0 & otherwise \end{cases} \qquad i = 1, 2, \ldots, n$$

For simplifying $y_{(i)}$ can be expressed by y_i. Since the lifetimes $y_1, \ldots, y_n$ of n items are independent and identically distributed random variables, then their likelihood function is given by

$$L(\underline{y};\beta,c,k)=\prod_{i=1}^{n}\{f_1(y_i)\}^{\delta_{1i}}\{f_2(y_i)\}^{\delta_{2i}}\{R(\eta)\}^{\bar{\delta}_{1i}\bar{\delta}_{2i}} \tag{3.1}$$

$$L\left(\underline{y};\beta,c,k\right)=\prod_{i=1}^{n}\left\{c\,k\,y_i^{-c-1}\left(1+y_i^{-c}\right)^{-(k+1)}\right\}^{\delta_{1i}}$$
$$\times\left\{c\,k\,\beta\left[\tau+\beta\left(y_i-\tau\right)\right]^{-c-1}\left[1+\left(\tau+\beta\{y_i-\tau\}\right)^{-c}\right]^{-(k+1)}\right\}^{\delta_{2i}}$$
$$\times\left\{1+\left[\tau+\beta\left(\eta-\tau\right)\right]^{-c}\right\}^{-k\,\bar{\delta}_{1i}\,\bar{\delta}_{2i}}$$

where, $\bar{\delta}_{1i}=1-\delta_{1i}$ and $\bar{\delta}_{2i}=1-\delta_{2i}$.

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is

$$\ln L=\sum_{i=n}^{n}\left\{\delta_{1i}\ln\left[c\,k\,y_i^{-c-1}\left(1+y_i^{-c}\right)^{-(k+1)}\right]\right\}$$
$$+\sum_{i=1}^{n}\left\{\delta_{2i}\ln\left[c\,k\,\beta\left(\tau+\beta\{y_i-\tau\}\right)^{-c-1}\left(1+\left[\tau+\beta\{y_i-\tau\}\right]^{-c}\right)^{-(k+1)}\right]\right\}$$
$$-k\sum_{i=1}^{n}\left\{\bar{\delta}_{1i}\bar{\delta}_{2i}\ln\left[1+\left(\tau+\beta\{\eta-\tau\}\right)^{-c}\right]\right\}$$

$$\ln L=\sum_{i=1}^{n}\delta_{1i}\ln c+\sum_{i=1}^{n}\delta_{1i}\ln k+\sum_{i=1}^{n}\delta_{1i}\left(c-1\right)\ln y_i-\left(k+1\right)\sum_{i=1}^{n}\delta_{1i}\ln\left(1+y_i^{-c}\right)$$
$$+\sum_{i=1}^{n}\delta_{2i}\ln c+\sum_{i=1}^{n}\delta_{2i}\ln k+\sum_{i=1}^{n}\delta_{2i}\ln\beta+\left(-c-1\right)\sum_{i=1}^{n}\delta_{2i}\ln\left[\tau+\beta\{y_i-\tau\}\right]$$
$$-\left(k+1\right)\sum_{i=1}^{n}\delta_{2i}\ln\left[1+\left(\tau+\beta\{y_i-\tau\}\right)^{c}\right]-k\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}\ln\left[1+\left(\tau+\beta\{\eta-\tau\}\right)^{-c}\right]$$

since $\sum_{i=1}^{n}\delta_{1i}=n_u$, $\sum_{i=1}^{n}\delta_{2i}=n_a$ and $\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}=n-n_u-n_a$, then

$$\ln L = n_u \ln c + n_u \ln k + \sum_{i=1}^{n} \delta_{1i} (-c-1) \ln y_i - (k+1) \sum_{i=1}^{n} \delta_{1i} \ln\left(1 + y_i^{-c}\right)$$
$$+ n_a \ln c + n_a \ln k + n_a \ln \beta + (-c-1) \sum_{i=1}^{n} \delta_{2i} \ln\left[\tau + \beta\{y_i - \tau\}\right]$$
$$-(k+1) \sum_{i=1}^{n} \delta_{2i} \ln\left[1 + \left(\tau + \beta\{y_i - \tau\}\right)^{-c}\right]$$
$$-k(n - n_u - n_a) \ln\left[1 + \left(\tau + \beta\{\eta - \tau\}\right)^{-c}\right]$$

$$\ln L = n_0 \ln c + n_0 \ln k + (-c-1)\left\{\sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln\left[\tau + \beta\{y_i - \tau\}\right]\right\}$$
$$+ n_a \ln \beta - (k+1)\left\{\sum_{i=1}^{n} \delta_{1i} \ln\left[1 + y_i^{-c}\right] + \sum_{i=1}^{n} \delta_{2i} \ln\left[1 + \left(\tau + \beta\{y_i - \tau\}\right)^{-c}\right]\right\} \quad (3.2) \, [$$
$$-k(n - n_0) \ln\left[1 + \left(\tau + \beta\{\eta - \tau\}\right)^{-c}\right]$$

where, $n_0 = n_u + n_a$

Maximum likelihood estimators of β, c and k are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood with respect to β, c and k be zero respectively. Therefore, the system of equations is as follows:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} + (-c-1) \sum_{i=1}^{n} \delta_{2i} (y_i - \tau)\left(\tau + \beta\{y_i - \tau\}\right)^{-1}$$
$$-(k+1)c \sum_{i=1}^{n} \delta_{2i} \left(\tau + \beta\{y_i - \tau\}\right)^{-c-1} (y_i - \tau)\left[1 + \left(\tau + \beta\{y_i - \tau\}\right)^{-c}\right]^{-1}$$
$$-k(n - n_0)\, c \left(\tau + \beta(\eta - \tau)\right)^{-c-1} (\eta - \tau)\left[1 + \left(\tau + \beta(\eta - \tau)\right)^{-c}\right]^{-1},$$

Let $A = [\tau + \beta\{y_i - \tau\}]$ and $D = [\tau + \beta(\eta - \tau)]$

then

$$\frac{\partial \ln L}{\partial \beta} = (-c-1) \sum_{i=1}^{n} \delta_{2i} (y_i - \tau)(A)^{-1} - k(n - n_0) c (D)^{c-1} (\eta - \tau)\left(1 + D^{c}\right)^{-1}$$
$$-(k+1)c \sum_{i=1}^{n} \delta_{2i} (A)^{-c-1} (y_i - \tau)\left(1 + A^{-c}\right)^{-1} + \frac{n_a}{\beta}, \quad (3.3)$$

$$\frac{\partial \ln L}{\partial c} = \frac{n_0}{c} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln(A) - k\,(n-n_0)(D)^{-c} \ln(D)\left(1+D^{-c}\right)^{-1}$$
$$-(k+1)\left\{\sum_{i=1}^{n} \delta_{1i}\, y_i^{-c} \ln y_i \left(1+y_i^{-c}\right)^{-1} + \sum_{i=1}^{n} \delta_{2i}\,(A)^{-c} \ln(A)\left(1+A^{-c}\right)^{-1}\right\}$$

(3.4) and

$$\frac{\partial \ln L}{\partial k} = \frac{n_0}{k} - \sum_{i=1}^{n} \delta_{1i} \ln\left(1+y_i^{-c}\right) - \sum_{i=1}^{n} \delta_{2i} \ln\left(1+A^{-c}\right) - (n-n_0)\ln\left(1+D^{-c}\right) \quad (3.5)$$

From equation (3.5) the maximum likelihood estimate of k is expressed by

$$\hat{k} = \frac{n_o}{a_1} \,. \quad (3.6)$$

where,

$$a_1 = \sum_{i=1}^{n} \delta_{1i} \ln\left(1+y_i^{-c}\right) + \sum_{i=1}^{n} \delta_{2i} \ln\left(1+A^{-c}\right) + (n-n_0)\ln\left(1+D^{-c}\right)$$

Consequently, by substituting for $\hat{k}$ into equation (3.3) and (3.4), the system equation are reduced into two nonlinear equation as follows:

$$\frac{n_0}{\hat{c}} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A - \left(\frac{n_0}{a_1}+1\right) a_2 - \left(\frac{n_0}{a_1}\right) a_3 = 0 \quad (3.7) \text{ and}$$

$$\frac{n_a}{\hat{\beta}} + (-\hat{c}-1)\sum_{i=1}^{n} \delta_{2i}\,(y_i-\tau)A^{-1} - \left(\frac{n_0}{a_1}+1\right) a_4 - \left(\frac{n_0}{a_1}\right) a_5 = 0 \quad (3.8) \text{ where,}$$

$$a_2 = \sum_{i=1}^{n} \delta_{1i} y_i^{-\hat{c}} \ln y_i \left(1+y_i^{-\hat{c}}\right)^{-1} + \sum_{i=1}^{n} \delta_{2i}\,(A)^{-\hat{c}} \ln A \left(1+A^{-c}\right)^{-1},$$

$$a_3 = (n-n_0)(D)^{-\hat{c}} \ln D \left(1+D^{-c}\right)^{-1},$$

$$a_4 = \hat{c}\sum_{i=1}^{n} \delta_{2i}\,(A)^{-\hat{c}-1}(y_i-\tau)\left(1+A^{-c}\right)^{-1}$$

and $a_5 = (n-n_0)\hat{c}\,(D)^{-\hat{c}-1}(\eta-\tau)\left(1+D^{-c}\right)^{-1}$.

Since the closed form solution to nonlinear equations (3.7) and (3.8) are very hard to obtain. An iterative procedure is applied to solve these equations numerically using Mathcad (2001) statistical package. Newton-Raphson method is applied for simultaneously solving the nonlinear equations to obtain $\hat{\beta}$ and $\hat{c}$. Therefore $\hat{k}$ is calculated easily from equation (3.6).

3.2 Interval Estimation

If $L_\theta = L_\theta(y_1,...,y_n)$ and $U_\theta = U_\theta(y_1,...,y_n)$ are functions of the sample data $y_1,...,y_n$ then a confidence interval for a population parameter θ is given by:

$$P[L_\theta \le \theta \le U_\theta] = \gamma, \qquad (3.9)$$

where, L_θ and U_θ are the lower and upper confidence limits which enclose θ with probability γ. The interval $[L_\theta, U_\theta]$ is called a 100γ % confidence interval for θ.

For large sample size, the maximum likelihood estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the approximate 100γ % confidence limits for the maximum likelihood estimate $\hat{\theta}$ of a population parameter θ can be constructed, such that

$$P[-z \le \frac{\hat{\theta}-\theta}{\sigma(\hat{\theta})} \le z\,] = \gamma, \qquad (3.10)$$

where, z is the $[\frac{100(1-\gamma)}{2}]$ standard normal percentile. Therefore, the approximate 100γ % confidence limits for a population parameter θ can be obtained, such that

$$P[\hat{\theta} - z\,\sigma(\hat{\theta}) \le \theta \le \hat{\theta} + z\,\sigma(\hat{\theta})] = \gamma, \qquad (3.11)$$

then the approximate confidence limits for β, c and k will be constructed using equation (3.11) with confidence levels 95 % and 99 %.

4. Asymptotic Variances and Covariance of Estimators

The asymptotic variances of maximum likelihood estimators are given by the elements of the inverse of the Fisher information matrix $I_{ij}(\underline{\theta}) = E\{-\partial^2 \ln L / \partial\theta_i \partial\theta_j\}$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the observed Fisher information matrix is given by $I_{ij}(\underline{\theta}) = \{-\partial^2 \ln L / \partial\theta_i \partial\theta_j\}$, which is obtained by dropping the expectation on operation E Cohen (1965). The approximate (observed) asymptotic variance covariance matrix F for the maximum likelihood estimates can be written as follows:

$$F = [I_{ij}(\underline{\theta})], \qquad i, j = 1,2,3 \quad \text{and} \quad (\underline{\theta}) = (c, \beta, k) \tag{4.1}$$

The second partial derivatives of the maximum likelihood function are given as follows:

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \beta^2} &= -\frac{n_a}{\beta^2} - (-c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)^2 (A)^{-2} \\ &\quad - (k+1)c\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)\Big[(-c-1)(y_i - \tau)(1 + A^{-c})^{-1}(A)^{-c-2} - c(y_i - \tau)(1 + A^{-c})^{-2}(A)^{-2c-2}\Big] \\ &\quad - k(n - n_0)c(\eta - \tau)\Big[(-c-1)(\eta - \tau)(1 + D^{c})^{-1}(D)^{-c-2} - c(\eta - \tau)(1 + D^{-c})^{-2}(D)^{-2c-2}\Big],\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial \beta \partial c} &= \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)(A)^{-1} \\ &\quad - k(n - n_0)(\eta - \tau)\Big[(D)^{-c-1}(1 + D^{-c})^{-1} + c(1 + D^{-c})^{-1}(D)^{c-1}\ln D - c(D)^{c-1}(1 + D^{-c})^{-2}(D)^{-c}\ln D\Big] \\ &\quad - (k+1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)\Big[c(1 + A^{-c})^{-1}(A)^{-c-1}\ln A + (A)^{-c-1}(1 + A^{-c})^{-1} - c(1 + A^{-c})^{-2}(A)^{-c-1}(A)^{-c}\ln A\Big],\end{aligned} \tag{4.3}$$

$$\frac{\partial^2 \ln L}{\partial \beta \partial k} = -c\sum_{i=1}^{n} \delta_{2i}(A)^{-c-1}(y_i - \tau)(1 + A^{-c})^{-1} - (n - n_0)c(D)^{-c-1}(\eta - \tau)(1 + D^{-c})^{-1}, \tag{4.4}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial c^2} &= -\frac{n_0}{c^2} - k(n - n_0)\ln D\Big[(1 + D^{-c})^{-1}(D)^{-c}\ln D - (1 + D^{-c})^{-2}(D)^{-2c}\ln D\Big] \\ &\quad - (k+1)\left[\sum_{i=1}^{n} \delta_{1i}\ln y_i\left\{(1 + y_i^{-c})^{-1} y_i^{-c}\ln y_i - (y_i^{-2c})(1 + y_i^{-c})^{-2}\ln y_i\right\}\right] \\ &\quad - (k+1)\left[\sum_{i=1}^{n} \delta_{2i}\ln A\left\{(1 + A^{-c})^{-1}(A)^{-c}\ln A - (1 + A^{-c})^{-2}(A)^{-2c}\ln A\right\}\right],\end{aligned} \tag{4.5}$$

$$\frac{\partial^2 \ln L}{\partial c\,\partial k} = -\sum_{i=1}^{n} \delta_{1i}\left(y_i^{-c}\right)\ln y_i\left(1+y_i^{-c}\right)^{-1} - \sum_{i=1}^{n} \delta_{2i}\left(A\right)^{-c}\ln A\left(1+A^{-c}\right)^{-1} - \left(n-n_0\right)\left(D\right)^{-c}\ln D\left(1+D^{-c}\right)^{-1}, \tag{4.6}$$

and

$$\frac{\partial^2 \ln L}{\partial k^2} = -\frac{n_0}{k^2}\ . \tag{4.7}$$

Consequently, the maximum likelihood estimators of β, c and k have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix F and by substituting $\hat{\beta}$, $\hat{c}$ and $\hat{k}$ for β, c and k respectively.

5. Simulation Studies

Simulation studies have been performed using Mathcad (2001) for illustrating the theoretical results of estimation problem. The performance of the resulting estimators of the acceleration factor and two shape parameters has been considered in terms of their absolute relative bias (ARBias); which is the absolute difference between the mean estimates and its true value divided by the true value of the parameter (*i.e.* $ARBias(\hat{\theta}) = \left|\frac{\hat{\theta}-\theta}{\theta}\right|$), mean square error (MSE); which is the sum squares of the difference between the estimated parameter and its true value divided by the number of the sample (*i.e.* $MSE(\hat{\theta}) = E\left[\left(\hat{\theta}-\theta\right)^2\right]$) , and relative error (RE); which is the square root of $MSE(\hat{\theta})$ divided by the value of the parameter (*i.e.* $RE(\hat{\theta}) = \frac{\sqrt{MSE(\hat{\theta})}}{\theta}$).

Furthermore, the asymptotic variance and covariance matrix and two-sided confidence intervals of the acceleration factor and two shape parameters are obtained. The Simulation procedures were described below:

Step 1: 1000 random samples of sizes 100 (50) 400 and 500 were generated from Burr type III distribution. The generation of Burr type III distribution is very simple, if U has a uniform (0,1) random number, then $Y = [U^{(\frac{-1}{k})} - 1]^{(\frac{-1}{c})}$ follows a Burr type III distribution. The true parameters values selected as $(c = 1.25, \beta = 1.5, k = 0.5)$ and $(c = 0.7, \beta = 1.15, k = 0.6)$

Step 2: Choosing the censoring time τ at the normal condition to be $\tau = 2$ and the total number of failure in the test of a PALT to be $r = 0.75n$.

Step 3: For each sample and for the two sets of parameters, the acceleration factor and the parameters of distribution were estimated in SS-ALT under type II censored sample.

Step 4. Newton Raphson method was used for solving the two nonlinear likelihood for β and c given in equations (3.7) and (3.8), respectively. The estimate of the shape parameter k was easily obtained from equation (3.6)

Step 5: The ARBias, MSE, and RE of the estimators for acceleration factor and two shape parameters for all sample sizes and for two sets of parameters were tabulated.

Step 6: The asymptotic variance and covariance matrix of the estimators for different sample sizes were obtained.

Step 7: The confidence limit with confidence level $\gamma = 0.95$ and $\gamma = 0.99$ of the acceleration factor and the two shape parameters were constructed.

Simulation results are summarized in Tables 1, 2 and 3. Table 1 gives the ARBias, MSE, and RE of the estimators. The asymptotic variances and covariance matrix of the estimators are displayed in Table 2. While Table 3 presents the approximated confidence limits at 95 % and 99% for the parameters and acceleration factor.

From these tables, the following observations can be made on the performance of SS-PALT parameter estimation of Burr type III lifetime distribution:

1. For the second set of parameters $(c = 0.7, \beta = 1.15, k = 0.6)$, the maximum likelihood estimators have good statistical properties than the first set of parameters $(c = 1.25, \beta = 1.5, k = 0.5)$ for all sample sizes (Table 1).

2. The asymptotic variances of the estimators are decreasing when the sample size increasing (Table 2).

3. The interval of the estimators decreases when the sample size is increasing. Also, the interval of the estimator at $\gamma = 0.95$ is smaller than the interval of estimator at $\gamma = 0.99$ (Table 3).

6. Conclusion

For products having a high reliability, the test of product life under normal use often requires a long period of time. So ALT or PALT is used to facilitate estimating the reliability of the unit in a short period of time. In OALT test items are run only at accelerated conditions, while in PALT they are run at both normal and accelerated conditions. One way to accelerate failure is the SS-PALT which increases the stresses applied to test product in a specified discrete sequence. The lifetime of the test items is assumed to follow the Burr type III distribution. Under type II censoring, the test unit is first run at normal use condition, and if it does not fail for a specified time τ, then it is run at accelerated condition until censoring number of failure r is reached. The maximum likelihood method is used for estimating the acceleration factor and parameters of Burr type III distribution under type II censoring. Performance of step-stress testing plans and model assumptions are usually evaluated by the properties of the maximum likelihood estimates of model parameters.

In this study, the second set of parameters have good statistical properties than the first set of parameters for all sample sizes maximum likelihood estimators are consistent and asymptotically normally distributed. As the sample size increases the asymptotic variance and covariance of estimators decreases. Regarding the interval of estimators, it can be noted that the interval of the estimators at $\gamma = 0.99$ is greater than the corresponding at $\gamma = 0.95$. Also, as the sample size increases the interval of the estimators decreases for the two confidence level.

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Table 1: The ARBias, MSE and RE of the Parameters (c,β,k,τ) given $r=0.75n$ for different sized samples Under Type II Censoring.

N	Parameters (c,β,k,τ)	$(1.25,1.5,0.5,2)$			$(0.7,1.15,0.6,2)$		
		ARBias	MSE	RE	ARBias	MSE	RE
100	c	0.0770	0.0330	0.1210	0.0320	0.0052	0.1040
	β	0.0870	0.0470	0.1450	0.0120	0.0160	0.1100
	k	0.1770	0.0110	0.2060	0.0780	0.0066	0.1350
150	c	0.0790	0.0270	0.1170	0.0290	0.0049	0.0910
	β	0.0870	0.0360	0.1270	0.0037	0.0110	0.0920
	k	0.1800	0.0100	0.2010	0.0840	0.0055	0.1240
200	c	0.0830	0.0250	0.1140	0.0270	0.0042	0.0870
	β	0.0890	0.0330	0.1210	0.0019	0.0081	0.0780
	k	0.1870	0.0100	0.2000	0.086	0.0049	0.1150
250	c	0.0850	0.0220	0.1100	0.0250	0.0041	0.0770
	β	0.0900	0.0300	0.1160	0.0032	0.0063	0.690
	k	0.1840	0.0096	0.1960	0.0890	0.0045	0.1120
300	c	0.0870	0.0190	0.0970	0.0210	0.0038	0.0690
	β	0.0940	0.0300	0.1160	0.0046	0.0051	0.0620
	k	0.1860	0.0095	0.1950	0.0890	0.0043	0.1090
350	c	0.0880	0.0170	0.0950	0.0180	0.0033	0.0650
	β	0.0960	0.0290	0.1140	0.0008	0.0044	0.0580
	k	0.1870	0.0095	0.1950	0.0860	0.0039	0.1040
400	c	0.0895	0.0150	0.0950	0.0170	0.0029	0.0630
	β	0.0940	0.0280	0.1100	0.0002	0.0041	0.0560
	k	0.1870	0.0094	0.1930	0.0880	0.0025	0.1040
450	c	0.0910	0.0140	0.0930	0.0130	0.0018	0.0610
	β	0.0960	0.0270	0.1110	0.0003	0.0033	0.0500
	k	0.1880	0.0093	0.1930	0.0900	0.0039	0.1040
500	c	0.0920	0.0120	0.0930	0.0120	0.0018	0.0610
	β	0.0950	0.0270	0.1090	0.0001	0.0032	0.0500
	k	0.1880	0.0093	0.1930	0.0911	0.0038	0.1040

Table 2: Asymptotic Variances and Covariance of Estimators Under Type II Censoring.

n	$(1.25, 1.5, 0.5, 2)$			$(0.7, 1.15, 0.6, 2)$		
	$\hat{c}$	$\hat{\beta}$	$\hat{k}$	$\hat{c}$	$\hat{\beta}$	$\hat{k}$
100	1.2160	-0.6700	-0.1540	0.7260	-0.0670	-0.0540
	-0.6700	0.0820	0.0160	-0.0670	0.0570	0.0007
	-0.1540	0.0160	0.0100	-0.0540	0.0007	0.0087
150	0.7370	-0.3730	-0.0440	0.5550	-0.0300	-0.0240
	-0.3730	0.0310	0.0029	-0.0300	0.0061	-0.0003
	-0.0440	0.0029	0.0042	-0.0240	-0.0003	0.0049
200	0.5880	-0.0850	-0.0210	0.4160	-0.0160	-0.0130
	-0.0850	0.0180	0.0006	-0.0160	0.0041	-0.0004
	-0.0210	0.0006	0.0026	-0.0130	-0.0004	0.0034
250	0.4470	-0.0460	-0.0150	0.2580	-0.0120	-0.0094
	-0.0460	0.0140	0.0003	-0.0120	0.0032	-0.0004
	-0.0150	0.0003	0.0019	-0.0094	-0.0004	0.0027
300	0.3407	-0.0350	-0.0120	0.1660	-0.0095	-0.0076
	-0.0350	0.0110	0.0001	-0.0095	0.0026	-0.0004
	-0.0120	0.0001	0.0016	-0.0076	-0.0004	0.0022
350	0.2960	-0.0280	-0.0094	0.1371	-0.0074	-0.0060
	-0.0280	0.0089	0.00006	-0.0074	0.0022	-0.0004
	-0.0094	0.00006	0.0013	-0.0060	-0.0004	0.0019
400	0.1770	-0.0230	-0.0075	0.1120	-0.0063	-0.0051
	-0.0230	0.0076	0.00004	-0.0063	0.0019	-0.0003
	-0.0075	0.00004	0.0011	-0.0051	-0.0003	0.0016
450	0.1590	-0.0200	-0.0067	0.0970	-0.0055	-0.0044
	-0.0200	0.0067	0.00003	-0.0055	0.0017	-0.0003
	-0.0067	0.00003	0.0010	-0.0044	-0.0003	0.0014
500	0.1370	-0.0180	-0.0058	0.0786	-0.0049	-0.0039
	-0.0180	0.0059	0.00004	-0.0049	0.0015	-0.0002
	-0.0058	0.00004	0.0009	-0.0039	-0.0002	0.0013

Table 3: Confidence Bounds of the Estimates at Confidence level at 0.95 and 0.99.

n	Parameters (c,β,k,τ)	$(1.25,1.5,0.5,2)$			$(0.7,1.15,0.6,2)$		
		Standard deviation	Lower Bound	Upper Bound	Standard deviation	Lower Bound	Upper Bound
100	c	0.231	0.852 0.866	1.264 1.349	0.052	0.426 0.417	0.915 0.936
	β	0.180	1.024 0.913	1.731 1.842	0.126	0.916 0.838	1.411 1.489
	k	0.053	0.308 0.275	0.516 0.548	0.066	0.423 0.382	0.683 0.724
150	c	0.203	0.969 0.910	1.343 1.403	0.042	0.443 0.420	0.921 0.933
	β	0.145	1.093 1.004	1.660 1.750	0.106	0.947 0.881	1.362 1.428
	k	0.044	0.324 0.297	0.496 0.523	0.054	0.443 0.409	0.656 0.690
200	c	0.175	0.988 0.936	1.311 1.362	0.038	0.471 0.463	0.933 0.954
	β	0.123	1.125 1.048	1.609 1.685	0.090	0.976 0.920	1.329 1.385
	k	0.036	0.337 0.314	0.476 0.499	0.046	0.458 0.429	0.639 0.667
250	c	0.152	1.007 0.962	1.289 1.334	0.037	0.590 0.561	0.775 0.805
	β	0.110	1.149 1.081	1.579 1.648	0.079	0.998 0.949	1.309 1.358
	k	0.033	0.344 0.323	0.472 0.492	0.041	0.466 0.440	0.628 0.653
300	c	0.131	1.019 0.978	1.274 1.314	0.034	0.598 0.570	0.772 0.799
	β	0.101	1.161 1.098	1.557 1.620	0.071	1.016 0.972	1.094 1.338
	k	0.029	0.350 0.332	0.463 0.481	0.038	0.473 0.449	0.620 0.644
350	c	0.093	1.023 0.984	1.264 1.302	0.032	0.600 0.574	0.761 0.786
	β	0.092	1.175 1.118	1.536 1.593	0.066	1.021 0.980	1.281 1.322
	k	0.028	0.352 0.335	0.461 0.478	0.035	0.479 0.457	0.618 0.640
400	c	0.072	1.032 0.997	1.253 1.288	0.029	0.603 0.578	0.759 0.783
	β	0.087	1.189 1.135	1.529 1.583	0.064	1.025 0.985	1.275 1.315
	k	0.025	0.358 0.343	0.455 0.471	0.032	0.484 0.464	0.610 0.630
450	c	0.059	1.037 1.003	1.249 1.283	0.027	0.607 0.585	0.749 0.772
	β	0.082	1.195	1.515	0.057	1.035	1.259

			1.144	1.566		1.000	1.294
	k	0.023	0.361 0.347	0.451 0.465	0.032	0.484 0.464	0.608 0.628
500	c	0.047	1.042 1.011	1.243 1.275	0.027	0.606 0.583	0.753 0.776
	β	0.078	1.205 1.157	1.509 1.557	0.058	1.036 1.001	1.264 1.300
	k	0.022	0.363 0.349	0.450 0.463	0.030	0.486 0.467	0.605 0.624

The first entire of each parameter is for 95 % significance level and the second for 99%

الاستدلال لمعالم توزيع بيير اعتمادا على العينات المراقبة في اختبارات الحياة المعجلة

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المستخلص: في اختبارات الحياة المعجلة، يتم تعريض الوحدات المراد اختبارها في ظل ظروف تشغيل أشد قسوة من الضغوط العادية التي تتعرض لها الوحدات، وذلك بهدف إنهاء التجربة بسرعة، والحصول على معلومات الصلاحية بالنسبة لهذه الوحدات المراد اختبارها.

ومن أمثلة ظروف التشغيل القاسية التي يتم تعريض الوحدات لها، درجة الحرارة العالية، والضغط الجوى، والتيار الكهربائي، والرطوبة ..الخ.

إن الهدف من هذه الدراسة هي تطبيق طريقة اختبارات الحياة المعجلة على توزيع بيير من النوع الثالث، بهدف تقدير معامل التعجيل acceleration factor ومعالم توزيع بيير للحياة وذلك باستخدام طريقة الإمكان الأكبر في ضوء أساليب التوقف السريع للتجربة، وهو

النوع الثاني من العينات المراقبة. ونظراً لأن حساب مقدرات المعالم بطريقة التقدير المستخدمة، يتطلب حل بعض المعادلات غير الخطية، فإنه تم استخدام الحزمة الإحصائية MathCAD لحل المعادلات غير الخطية لتنفيذ طريقة التقدير. أيضاً، تم إيجاد مقدرات الفترة لمعالم توزيع بيير، وكذلك معامل التعجيل. وتم دراسة خصائص هذه المقدرات بالحصول على مصفوفة التباين والتغاير للمقدرات. وأخيرا تم استخدام أسلوب المحاكاة للوقوف على أهمية واستخدامات النتائج المتحصل عليها بالدراسة.