Discrete Adomian Decomposition Solution of Nonlinear Mixed Integral Equation

F. A. Hendi and H.O. Bakodah

Department of Mathematics, Science Faculty for Girls, King Abdulaziz University, Jeddah, Saudi Arabia
falhendi@kau.edu.sa; hbakodah@kau.edu.sa

Abstract: The main purpose of this paper is to use discrete Adomian decomposition method for solving mixed nonlinear Volterra-Fredholm integral equations of the second kind. This method is based upon quadrature rule and Adomian decomposition method. Numerical illustrations are investigated to show features of the technique.

[F. A. Hendi and H.O. Bakodah **Discrete Adomian Decomposition Solution of Nonlinear Mixed Integral Equation]** Journal of American Science 2011; 7(12):1081-1084]. (ISSN: 1545-1003). http://www.americanscience.org.

Keywords: Mixed Integral equation; Discrete Adomian decomposition method (DADM), quadrature rule.

2000 Mathematics subject classification: 45F05, 65R20, 65D32

Introduction

In recent years, the theory of integral equations has close contacts with many different areas of mathematics. For this, the analytic and numerical method play an important role for solving integral equations. The mixed Volterra-Fredholm integral equations arise in the theory of parabolic boundary value problems, the mathematical modeling of spatio-temporal development of an epidemic and various physical and biological problems. A discussion of the formulation of these models is given in **Wazwaz** [1] and the references therein. The nonlinear mixed Volterra-Fredholm integral equation is given in [1] as

$$u(x,t) = f(x,t) + \int_0^t \int_\Omega F(x,t,s,\tau,u(s,\tau))\,ds\,d\tau \quad ,(x,t) \in [0,T] \times \Omega \quad (1)$$

where $u(x,t)$ is an unknown function.

The functions $f(x,t)$ and $F(x,t,s,\tau,u(s,\tau))$ are analytical on $D := [0,T] \times \Omega$ and $(S \times R)$ where ($S = \{(x,t,s,\tau): 0 \le \tau \le t \le T, (x,s) \subset \Omega \times \Omega\}$). The existence and uniqueness results for Eq.(1) may be found in [2, 3] (see also [4, 5] for the linear case). The literature of integral equations contains few numerical methods for handling the Volterra-Fredholm integral equation (1). For the linear case, some projection methods for numerical treatment of (1) are given in [2, 4- 6]. The time collocation method was introduced in (1986) by **Pachpatta** [2] and the projection method [4] was presented in **Hacia** (1996). **Kauthen** [5] studied continuos time collocation, time discretization collocation methods, and analyzed their global discrete convergence properties local and global super convergence properties. The results of **Kauthen** have been extended to nonlinear Volterra-Fredholm integral equations by **Brunner** [3]. Then, **Guoqiang** [6] considered the particular trapezoidal Nystrom method for Eq.(1), and gave the asymptotic error expansion. **Maleknejad and Hadizadeh** [7] treated Eq.(1) by using the standard version of Adomian decomposition method [8, 9]. The modified decomposition method for (1) was given by **Wazwaz** [1]. The Adomian decomposition scheme is a method for solving a wide range of problems whose mathematical models yield equation or system of equations involving algebraic, differential, integral and integro-differential equations[8, 9]. In this work, we are concerned with the numerical solution of Eq.(1) by means of discrete Adomian decomposition method, which introduced by **Behiry *et al.*** [10]. (DADM) arises when the quadrature rules are used to approximate the definite integrals which cannot be computed analytically in Adomian decomposition method. The (DADM) gives the numerical solution at nodes used in quadrature rules. In [10] **Behiry *et al.*** applied the (DADM) to nonlinear Fredholm integral equation, this approach shows that the (DADM) leads to very significant improvement in accuracy comparing with standard version of Adomian decomposition method (ADM). For this, we use (DADM) to obtain numerical solutions of nonlinear mixed Volterra-Fredholm integral equations.

2Adomian Decomposition Method (ADM) for Mixed Volterra-Fredholm Integral Equations

Consider the Volterra-Fredholm integral equation (1). The Adomian decomposition method introduces the following expression

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \qquad (2)$$

for the solution $u(x,t)$ of (1), where the components $u_n(x,t)$ will be determined recurrently. Moreover, the method defines the nonlinear function $F(x,t,s,\tau,x(s,\tau))$ by an infinite series of polynomials

$$F(x,t,s,\tau,x(s,\tau)) = \sum_{n=0}^{\infty} A_n(t) \qquad (3)$$

where A_n are so-called Adomian polynomials that represent the nonlinear term $F(x,t,s,\tau,u(s,\tau))$ and can be calculated for various classes of nonlinear operators according to specific algorithms set by Adomian [8, 9]. Substituting (2) and (3) into (1) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x,t) + \int_0^t \int_\Omega \left(\sum_{n=0}^{\infty} A_n \right) ds d\tau \qquad (4)$$

It is import to note that the decomposition method suggests that the zeroth component u_0 is usually identified by the function f described above. As a result, the decomposition method introduces the recursive relation

$$u_0(x,t) = f(x,t) \qquad (5)$$

$$u_{K+1}(x,t) = \int_0^t \int_\Omega A_n ds d\tau \qquad ,K \geq 0 \qquad (6)$$

Having determined the components $u_0, u_1, u_2, \ldots$ the solution u in a series form defined by (2) follows immediately.

Generally, a complicated term $f(x,t)$ in a nonlinear equation (1), can causes difficult integrations and proliferation of terms in the Adomian recursive scheme, furthermore owing to the large variety of kernels (e.g. convolution, weakly singular,…) and nonlinearities that occur in practice, leading to difficult computations of the A_n polynomials and Adomian scheme. In [7] **Maleknejad and Hadizadeh** worked with a bound for the Adomian decomposition series, without computation of each term of the algorithm. They gave a bound for the Adomian decomposition series by the theorem 2.1

Theorem 2.1

Suppose that

(i) $G(x,t,\tau,u) = \int_\Omega F(x,t,s,u(s,\tau))ds;$

(ii) There exist nonnegative continuous $m(x,t)$ and $n(\tau)$ defined on D and R, respectively, such that $G(x,t,\tau,u)$ satisfies to a generalized Lipschitz condition of the form

$$\|G(x,t,\tau,u_1) - G(x,t,\tau,u_2)\| \leq m(x,t)n(\tau)\|u_1 - u_2\|$$

Then, the bound for the Adomian decomposition series for (1), can be established as

$$\left\| \sum_{K=0}^{\infty} A_n \right\| \leq v(x,t) + m(x,t) \int_0^t v(x,t)n(\tau) exp\left[\int_\tau^t m(x,\eta)n(\eta)d\eta \right] d\tau \qquad (7)$$

Where

$$v(x,t) = \left\| \int_0^t G(x,t,\tau,f(x,t))d\tau \right\|$$

Maleknejad and Hadizadeh presented numerical examples to illustrate the implementation and accuracy of the decomposition method and gave the efficiency and effectiveness of this bound. To overcome the tedious work of the existing strategies, the discrete Adomian decomposition method developed by **Behiry *et al.*** [10] will form a useful method for solving the Volterra-Fredholm integral equation. In section 3 we use the discrete Adomian decomposition method to obtain numerical solution of nonlinear mixed Volterra integral equations.

3 The (DADM) Applied to Volterra-Fredholm Integral Equations

While the (ADM) has been proved to be effective, there has been cumbersome work conducted in [7] and a numerical approximation was developed. It was observed in [9] that a complicated term $f(x,t)$ can cause difficult integrations and proliferation of terms in Adomian recursive scheme. If the evaluation of integrals analytically is possible, the (ADM) can be applied. In the cases where the evaluation of integral (6) is analytically impossible, the (ADM) cannot be applied. So we consider any numerical integration scheme be given by the following formula

$$\int_a^b g(s)ds \approx \sum_{j=0}^{n} w_{n,j} g(s_{n,j}) \qquad (8)$$

where $g(s)$ is continuous function on Ω , $s_{n,j}$ are the nodes of the quadrature rule and $w_{n,j}$ are the weights functions. Applying formula (8) on Eq. (6) to obtain

$$u(x,t) - f(x,t) + \int_0^t \int_\Omega G(x,t,s,\tau)(1 - e^{-u(s,\tau)}) ds d\tau \ ,$$

$$u_{K+1}(x,t) = \int_0^t \left(\sum_{\substack{j=0 \\ K\geq 0}}^{n} W_{n,j} A_K(x,t,s_{n,j},\tau,x(s_{n,j},\tau)) \right) d\tau \qquad (9)$$

The approximate solution of Eq.(1) using (DADM) can be obtained by summing the approximate values to the component $u_m(x,t), m \geq 0$ represented by Eqs. (5) and (9) at nodes $x_i, t_i, i = 0,1,2,\dots,n$ which are the some points of quadrature rule. The solution $u(x_i,t_i)$ at these nodes using (DADM) of Eq. (1) can be written as

$$u(x_i,t_i) = \sum_{m=0}^{\infty} u_m(x_i,t_i) \qquad (10)$$

4 Numerical Examples

In this section, we use the method discussed of the previous section for solving some examples.

Example 1

Consider the nonlinear mixed integral equation [12]

$$u(x,t) = xt - e^t + t + 1 + \int_0^t \int_0^1 te^{u(s,t)} ds dt \quad , \quad 0 \leq t \leq 1 \qquad (11)$$

with the exact solution $u(x,t) = xt$.

The approximate solution u_n with $n = 8$ and the absolute error $|e_n| = |u_e - u_n|$ are shown in Table (1).

Table (1) The results of Example (1), m=4

t	x	$\lvert e_8 \rvert$
0.00	0.00	0.00
0.125	0.125	1.04925E-9
0.250	0.250	3.21068E-7
0.375	0.375	9.21068E-6
0.500	0.500	1.02806E-4
0.625	0.625	6.59589E-4
0.750	0.750	2.97023E-2
0.875	0.875	1.03695E-2
1.00	1.00	2.97745E-2

Example 2

Consider the nonlinear Volterra-Fredholm integral equation

$$(x,t) \in \Omega \times [0,T] \qquad (12)$$

where

$$f(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right) + \frac{xt^2}{8(1+t)(1+t^2)}$$

with $\Omega = [0,1]$. The exact solution $u(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right)$.

The approximate solution u_n with $n = 8$ and the absolute error $|e_n| = |u_e - u_n|$ are shown in Table (2).

Table (2) the results of Example (2), m=4

t	x	$\lvert e_8 \rvert$
0.00	0.00	0.00
0.125	0.125	1.65481E-8
0.250	0.250	4.25697E-7
0.375	0.375	2.51102E-6
0.500	0.500	7.99106E-6
0.625	0.625	1.80488E-5
0.750	0.750	3.28240E-5
0.875	0.875	5.15273E-5
1.00	1.00	7.28514E-5

5 Conclusion

Many applications lead to the mixed nonlinear Volterra-Fredholm integral equation (1). Few numerical method have been used for solve this equation using an efficient numerical method could make easy to solve equations which are usually difficult for solving analytically with giving an approximate solution. In this paper, using(DADM) for solving a nonlinear mixed Volterra-Fredholm integral equation. (ADM) combined with quadrature rule is implemented in a straightforward manner and provided significant improvement in obtaining numerical solution by using few iterations. Moreover, the Adomian polynomials, that may cause difficult integrations in Adomian scheme, were computed by quadrature rule and this gives the (DADM) an advantage over the (ADM).

Corresponding author

F. A. Hendi
H.O. Bakodah
Department of Mathematics, Science Faculty for Girls, King Abdulaziz University, Jeddah, Saudi Arabia
falhendi@kau.edu.sa; hbakodah@kau.edu.sa

References

[1] Wazwaz A. M. (2002). A reliable treatment for mixed Volterra-Fredholm integral equations with applications, Appl. Math. Comput., 127: 405–414.

[2] Pachpathe B.G. (1986). On mixed Volterra-Fredholm type integral equations, Ind. J. pure Appl. Math. 17: 488-496.
[3] Brunner H. (1990). On the numerical solution of nonlinear Volterra-Fredholm integralequation by collocation methods, SIAM J. Numer. Anal., 27(4):987-1000.
[4] Hacia L. (1996). On approximate solution for integral equations of mixed type, ZAMM. Z. Angew. Math. Mech., 76: 415-416.
[5] Kauthen P.J. (1989). Continuous time collocation methods for Volterra-Fredholm integral equations, Numer. Math., 56:409-424 .
[6] Guoqiang H. (1995). Asymptotic error expansion for the Nyström method for a nonlinear Volterra-Fredholm integral equations, J. Comput. Math. Appl., 59:49.
[7] Maleknejad K., M. Hadrzaden (1999). A new computational method for Volterra-Fredholm integral equations, J. Comput. Math. Appl., 37:1-8.
12/2/2011
[8] Adomian G. (1994). Solving Frontire problems of Physics: The Decomposition Method, Dordrecht, kluwer,.
[9] Adomian G. (1990). A Review of the decomposition method and some recent results for nonlinear equations, Math. Comput. Model, 13:17-43.
[10] Behiry S.H., R.A. Abd-Elmonem and A.M. Gomaa (2010). Discrete Adomian decomposition solution of nonlinear Fradholm integral equation, Ain shams Engineering Journal, 1:97-101.
[11] Du H. and M. Cui (2007). Amethod of solving nonlinear mixed Volterra-Fredholm integral equation, Appl. Math. So. Vol.1, no.(51):2505-2516.
[12] Jafari H. and M.A. Firoozjace (2010). Multistage Homotopy analysis method for solving nonlinear integral equations, Appl. Math. No.(1) 34-45.