METRIC SPACES

1. Metric Spaces

Definition 1.1:[Metric Space]

Let X be a none-empty set. A mapping $d : X \times X \longrightarrow \mathbb{R}$ is said to be a metric space on x if it satisfies the following conditions

(1) $d(x,y) \geq 0 \quad \forall x,y \in X$

(2) $d(x,y) = 0 \Leftrightarrow x = y \quad \forall x,y \in X$

(3) $d(x,y) = d(y,x), \quad \forall x,y \in X$ (Symmetric)

(4) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$ (Triangle Inequality)

Example 1: Let X be any non-empty set .Define a mapping $d : X \times X \longrightarrow \mathbb{R}$ by $d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$

Then d is a metric on X, and this metric is called discrete metric.

Example 2: Let $X = \mathbb{F}^n(\mathbb{C}^n, \mathbb{R}^n), \quad n \geq 1$. Let $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in \mathbb{F}^n$ Define $d : X \times X \longrightarrow \mathbb{R}$ by $d(x,y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2}$. To see the Triangle Inequality, we will need **Minkowski Inequality** If $a_i, b_i \in \mathbb{F} \quad i = 1, 2, \cdots, n$, and $1 < p < \infty$, then

$$\sqrt[p]{\sum_{i=1}^{n} |a_i + b_i|^p} \leq \sqrt[p]{\sum_{i=1}^{n} |a_i|^p} + \sqrt[p]{\sum_{i=1}^{n} |b_i|^p}$$

Now, let $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n), z = (z_1, z_2, \cdots, z_n) \in \mathbb{F}^n$

$$\begin{aligned} d(x,y) &= \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2} && \text{Add and Subtract } z_k. \\ &= \sqrt{\sum_{k=1}^{n} |(x_k - z_k) + (z_k - y_k)|^2} && \text{Use Minkowski Inequality .} \\ &\leq \sqrt{\sum_{k=1}^{n} |x_k - z_k|^2} + \sqrt{\sum_{k=1}^{n} |z_k - y_k|^2}. \\ &= d(x,z) + d(z,y) \end{aligned}$$

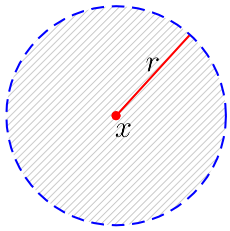

FIGURE 1

Example 3: l_p. Let p be a real number such that $1 \leq p < \infty$. l_p is the space of all sequence $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{F}$ such that $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty \quad (x = \{x_n\}_{n=1}^{\infty}$ converges).

$l_p = \{x = \{x_n\}_{n=1}^{\infty} \mid \sum\limits_{n=1}^{\infty} |x_n|^p < \infty, \quad x_n, \in \mathbb{F}, \forall n \in \mathbb{N}\}$

Define the mapping $d : l_p \times l_p \longrightarrow \mathbb{R}$ by $d(x,y) = d(\{x_n\},\{y_n\}) = \sqrt[p]{\sum\limits_{n=1}^{\infty} |x_n - y_n|^p}$. Then d is a metric on l_p.

Example 4: $C([a,b])$. Let a,b be two real numbers such that $a < b$. $C([a,b])$ is the space of all continuous real-valued functions f over $[a,b]$.

$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f$ is continuous on $[a,b]\}$ Define the mappings $d_1, d_\infty : C([a,b]) \times C([a,b]) \longrightarrow \mathbb{R}$ as follows:

$d_\infty(f,g) = \sup\limits_{x \in [a,b]} |f(x) - g(x)|$ and

$d_1(f,g) = \int\limits_a^b |f(x) - g(x)|\,dx$. Then d_1, d_∞ are metrics on $C([a,b])$.

Open sets and closed sets.

Definition 1.2:[Basic Definition]

Let (X,d) be a metric space. Let $E \subseteq X$, and let $x_0 \in X$.

(1) Let $x \in X$ and $r > 0$. We define the *open ball* of radius r about x to be the set $B_r(x) = \{y \in X \mid d(x,y) < r\}$.

(2) We say that E is open set if for each $x \in E$ there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq E$.

(3) We say that E is closed set if $E^c = X \setminus E$ is open set.

(4) We say that x_0 is an interior point of E if there exist $r > 0$ such that $B_r(x_0) \subseteq E$.

(5) We say that x_0 is a limit point of E if for each $r > 0$, $B_r(x_0) \cap (E \setminus \{x_0\}) \neq \phi$.

(6) The set of all interior points of E is denoted by E°.

(7) The set of all limit points of E is denoted by E'.

(8) The closure set of E ,denoted by $\overline{E}$, is $\overline{E} = E \cup E'$.

(9) We say that x_0 is a boundary point of E if for each $r > 0$, $B_r(x_0) \cap E \neq \phi$ and $B_r(x_0) \cap E^c \neq \phi$.

(10) The set of all boundary points of E is denoted by ∂E.

Below you will find some of elementary results about metric space- you should try to prove them-

Result 1.1:

Let (X,d) be a metric space. Let $E \subseteq X$. Then

(1) ϕ and X are both open and closed.

(2) An arbitrary union of open sets in X and a finite intersection of open sets in X are open sets in X.

(3) An arbitrary intersection of closed sets in X and a finite union of closed sets in X are closed sets in X.

(4) G is open $\Leftrightarrow G = G^{\circ}$.

(5) G is closed $\Leftrightarrow G = \overline{G}$.

Definition 1.3:[Distance between sets and diameter]

Let (X,d) be a metric space. Let $F, E \subseteq X$, and let $x_0 \in X$.

(1) The distance between E and $x_0 \in X$ is denoted by $D(x_0,E)$, is defined by $D(x_0,E) = \inf_{y \in E} d(x_0,y)$.

(2) The distance between the sets E and F, denoted by $D(F,E)$, is defined as $D(F,E) = \inf_{x \in F, y \in E} d(x,y)$.

(3) The diameter of the set F, denoted by $\delta(F)$, is defined as $\delta(F) = \sup_{x,y \in F} d(x,y)$.

Result 1.2:

Let (X,d) be a metric space. Let $E, F \subseteq X$. Then

(1) $D(\overline{F},\overline{E}) = D(E,F)$.

(2) If $x \in \overline{E} \Leftrightarrow D(x,E) = 0$.

(3) $\delta(\overline{E}) = \delta(E)$.

(4) If $E \subseteq F \Rightarrow \delta(E) \leq \delta(F)$.

(5) $E = \{x\} \Leftrightarrow \delta(E) = 0$.

(6) Let $x, y \in X$, then $|D(x,E) - D(y,F)| \leq d(x,y)$.

Convergence and completeness.

Definition 1.4:[Convergent and Cauchy Sequence]

Let (X,d) be a metric space. Let $\{x_n\} \subseteq X$, be a sequence.

(1) We say that $\{x_n\}$ is convergent to $x \in X$ if for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that if $n > N \Rightarrow d(x_n,x) < \varepsilon$, and we write $\lim_{n \to \infty} x_n = x$.

(2) We say that $\{x_n\}$ is Cauchy if for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that if $n,m > N \Rightarrow d(x_n,x_m) < \varepsilon$.

(3) We say that $\{x_n\}$ is bounded if there exist $M > 0$ such that $\delta(\{x_n\}) \leq M$.

Result 1.3:

Let (X,d) be a metric space. Let $\{x_n\} \subseteq X$, be a sequence. Then

(1) If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded and its limit is unique.

(2) A Cauchy sequence is bounded.

(3) A convergent sequence is Cauchy.

(4) If $x_n \longrightarrow x, y_n \longrightarrow y$ in $X \Rightarrow d(x_n,y_n) \longrightarrow d(x,y)$.

(5) If $\{x_n\}$ is convergent to $x \in X$, then every subsequence $\{x_{n_k}\}$ is convergent to $x \in X$.

(6) If $\{x_n\}$ is Cauchy sequence and if $\{x_{n_k}\}$ is convergent subsequence to $x \in X$, then $\lim_{n\to\infty} x_n = x$.

Definition 1.5:[Complete Metric]

Let (X,d) be a metric space. We say that X is complete metric space if every Cauchy sequence in X converges to a point in X.

Example 5:

(1) The spaces $\mathbb{R}^n$, $\mathbb{C}^n$ with $d(x,y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$ are complete.

(2) The space l_p with the metric $d(x,y) = d(\{x_n\},\{y_n\}) = \sqrt[p]{\sum_{n=1}^{\infty} |x_n - y_n|^p}$ is complete.

(3) The space l_∞ with the metric $d(x,y) = d(\{x_n\},\{y_n\}) = \sup_n |x_n - y_n|$ is complete.

(4) The space $C([a,b])$ with the metric $d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$ is complete.

Example 6:

(1) The spaces $\mathbb{Q}$ with $d(x,y) = |x-y|$ is not complete. For example the sequence $x_1 = 1, x_{n+1} = \dfrac{x_n^2+2}{2x_n}$ is in $\mathbb{Q}$, but $\lim_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}$.

(2) The space $C([-1,1])$ with the metric $d_1(f,g) = \int_{-1}^{1} |f(x) - g(x)|\,dx$. is not complete. For example let

$$f_n(x) = \begin{cases} 1, & \text{if } -1 \le x \le 0; \\ 1-nx, & \text{if } 0 < x < 1/n; \\ 0, & \text{if } 1/n \le x \le 1. \end{cases}$$

. Then $\{f_n\}$ is Cauchy but it converges $\chi_{[-1,0]} \notin C([-1,1])$

Theorem 1.1: []

Let E be a subset of a metric space (X,d) and let $x \in X$. Then

(1) $x \in \overline{E} \Leftrightarrow \exists \{x_n\} \subset E \ni \lim_{n\to\infty} x_n = x$.

(2) $x \in E\prime \Rightarrow \exists \{x_n\} \subset E \ni \lim_{n\to\infty} x_n = x$.

Proof:

(1) ($\Rightarrow$) Suppose that $x \in \overline{E}$. For each $n \in \mathbb{N}$, $B_{1/n}(x) \cap E \neq \phi$. Pick $x_n \in B_{1/n}(x) \cap E$. Now, $d(x_n, x) < \frac{1}{n} \Rightarrow \lim_{n\to\infty} x_n = x$.
($\Leftarrow$) If $\{x_n\} \subset E \ni \lim_{n\to\infty} x_n = x$, then for any $r > 0$, $\exists \quad N \in \mathbb{N} \ni$ if $n > N \Rightarrow d(x, x_n) < r$. Thus $B_r(x) \cap E \neq \phi$.
Hence $x \in \overline{E}$.

(2) Suppose thst $x \in E\prime$. Choose $x_1 \in E$ such that $x_1 \neq x$ and $d(x, x_1) < 1$. Now, choose $x_2 \in E \setminus \{x\}$ such that $d(x, x_2) < \min\{d(x, x_1), 1/2\}$. Now, for each $n \geq 3$ pick $x_n \in E \setminus \{x\}$ such that $d(x, x_n) < \min\{d(x, x_{n-1}), 1/n\}$. Hence we have a sequence $\{x_n\} \subset E$ such that $\lim_{n\to\infty} x_n = x$.

Dense sets and Separable Spaces.

Definition 1.6:[Separable Spaces]

Let (X, d) be a metric space. Let $E \subseteq X$.

(1) We say that E is dense if $\overline{E} = X$.

(2) We say that X is separable if it has a countable dense subset.

Example 7:

(1) The spaces $\mathbb{R}^n$, $\mathbb{C}^n$ with $d(x, y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$ are separable.

(2) The space l_p with the metric $d(x, y) = d(\{x_n\}, \{y_n\}) = \sqrt[p]{\sum_{n=1}^{\infty} |x_n - y_n|^p}$.

(3) The space l_∞ with the metric $d(x, y) = d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|$ is not separable.

(4) The space $C([a, b])$ with the metric $d_\infty(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|$ is separable.

Continuous Functions.

Definition 1.7:[Continuity]

Let $(X, d_1), (Y, d_1)$ be a metric spaces. A function $f : X \longrightarrow Y$ is called continuous at $x_0 \in X$ if for each $\varepsilon > 0$, $\exists \quad$ a $\delta > 0$ such that $x \in X$ and $d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon$. f is continuous on X if it is continuous at each point of X.

Theorem 1.2: []

Let $(X, d_1), (Y, d_1)$ be a metric spaces and let $f : X \longrightarrow Y$ be a function Then, the following statements are equivalent:

(1) f is continuous on X.

(2) $\forall x \in X$ and $\forall \quad \{x_n\} \subset X \ni \lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x)$.

(3) For each E is open in $Y \Rightarrow f^{-1}(E)$ is open in X

(4) For each F is closed in $Y \Rightarrow f^{-1}(E)$ is closed in X

(5) $f(\overline{E}) \subset \overline{f(E)}, \forall \quad E \subseteq X$.

(6) $\overline{f^{-1}(F)} \subset f^{-1}(\overline{F}), \forall \quad F \subseteq Y$.

Definition 1.8:[Homeomorphism]

Let $(X,d_1),(Y,d_1)$ be a metric spaces.

(1) A function $f: X \to Y$ is called homeomorphism if f is a continuous bijective and f^{-1} is continuous.

(2) A function $f: X \to Y$ is called isometry if $d_2(f(x), f(y)) = d_1(x,y), \qquad x,y \in X$.