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0.5 Transformations of Functions

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Combinations of Functions

Algebra of Functions:

Definition .1: [The Sum, Difference, Product, and Quotient of two functions]
Let f and g be functions with domains $D(f)$ and $D(g)$. Then the functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ are defined as follows:

Name	Notation	Definition	Domain
Sum:	$f+g$	$(f+g)(x) = f(x)+g(x)$	$D(f)\cap D(g)$
Difference:	$f-g$	$(f-g)(x) = f(x)-g(x)$	$D(f)\cap D(g)$
Product:	fg	$(fg)(x) = f(x)g(x)$	$D(f)\cap D(g)$
Quotient:	$\frac{f}{g}$	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	$(D(f)\cap D(g))\setminus\{x : g(x)=0\}$

EXAMPLE 1. Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{2x-4}$. Find $f+g$, $f-g$, fg, f/g, and their domains.

Solution: Since $f(x) = x^2 + 1$ is a polynomial, then $D(f) = \mathbb{R}$. Also, since $g(x) = \sqrt{2x-4}$ is an even root function, then to find the domain we solve the inequality

$$2x - 4 \geq 0 \Leftrightarrow 2x \geq 4 \Leftrightarrow x \geq 2.$$

Hence $D(g) = [2, \infty)$. Now, $D(f) \cap D(g) = \mathbb{R} \cap [2, \infty) = [2, \infty)$.

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \text{By definition}\\
&= x^2 + 1 + \sqrt{2x-4} && \text{With } D(f+g) = [2,\infty),\\
(f-g)(x) &= f(x) - g(x) && \text{By definition}\\
&= x^2 + 1 - \sqrt{2x-4} && \text{With } D(f-g) = [2,\infty),\\
(fg)(x) &= f(x)g(x) && \text{By definition}\\
&= (x^2+1)\sqrt{2x-4} && \text{With } D(fg) = [2,\infty),\\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} && \text{By definition}\\
&= \frac{x^2+1}{\sqrt{2x-4}} && \text{With } D(f+g) = [2,\infty) \setminus \{x : 2x-4 = 0\},\\
& && \text{With } D(f+g) = [2,\infty) \setminus \{2\} = (2,\infty).
\end{aligned}$$

□

EXAMPLE 2. Let $f(x) = x + \sqrt{x^2 - 1}$, $g(x) = \dfrac{2x\sqrt{4 - x^2}}{x+1}$ and $h(x) = \dfrac{\sqrt{2x-1}}{\sqrt[4]{x^2 - 5x + 6}}$. Find the domain of f, g and h.

Solution:

- Notice that $f(x) = x + \sqrt{x^2 - 1}$ is the sum of x and $\sqrt{x^2 - 1}$, then
$$D(f) = D(x)^{?} \cap D(\sqrt{x^2 - 1})^{?} = \mathbb{R} \cap (-\infty - 1] \cup [1, \infty) = (-\infty - 1] \cup [1, \infty).$$

- Notice that $g(x) = \dfrac{2x\sqrt{4 - x^2}}{x+1} = \dfrac{2x}{x+1} \cdot \sqrt{4 - x^2}$ is the product of $\frac{2x}{x+1}$ and $\sqrt{4 - x^2}$, then
$$D(g) = D(\tfrac{2x}{x+1})^{?} \cap D(\sqrt{4 - x^2})^{?} = (-\infty, -1) \cup (-1, \infty) \cap [-2, 2] = [-2, -1) \cup (-1, 2].$$

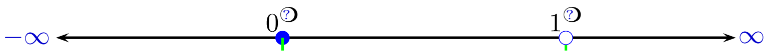
- Notice that $h(x) = \dfrac{\sqrt{2x-1}}{\sqrt[4]{x^2 - 5x + 6}}$ is the quotient of $\sqrt{2x - 1}$ and $\sqrt[4]{x^2 - 5x + 6}$, then
$$\begin{aligned} D(h) &= D(\sqrt{2x-1})^{?} \cap D(\sqrt[4]{x^2 - 5x + 6})^{?} \setminus \{x : x^2 - 5x + 6 = 0\} \\ &= [1/2, \infty) \cap (-\infty, 2] \cup [3, \infty) \setminus \{2, 3\} \\ &= [3, \infty) \setminus \{2, 3\} = (3, \infty). \end{aligned}$$

□

Example 3. Let $g(x) = \sqrt{\dfrac{x}{x-1}}$. Find the domain of g.

Solution:

The function $g(x) = \sqrt{\dfrac{x}{x-1}}$ is an even root function, hence $\dfrac{x}{x-1} \geq 0$ (we are looking for (+) sign). Now, to solve the inequality $\dfrac{x}{x-1} \geq 0$, we find the zeros of the numerator and the denominator. $x = 0$ is the zero of numerator and $x - 1 = 0 \Leftrightarrow x = 1$ is the zero of the denominator. Now, we use the real line to find the sign of each expression x and $x - 1$.



Test Value	-1	$.5$	2
sign of x	$-1 = -1$ $(-)$	$0.5 = 0.5$ $(+)$	$2 = 2$ $(+)$
sign of $x - 1$	$-1 - 1 = -2$ $(-)$	$0.5 - 1 = -0.5$ $(-)$	$2 - 1 = 1$ $(+)$
sign of $\frac{x}{x-1}$	$(-)/(-) = +$	$(+)/(-) = -$	$(+)/(+) = +$

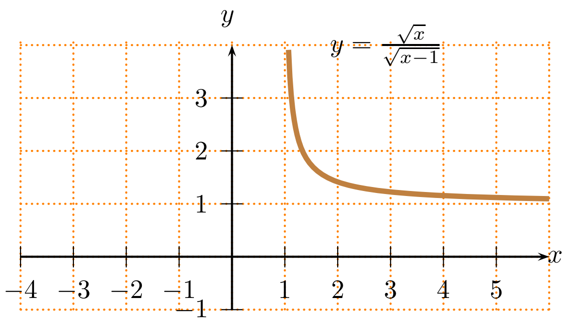
Hence $D(g) = (-\infty, 0] \cup (1, \infty)$.

□

EXAMPLE 4. Let $f(x) = \dfrac{\sqrt{x}}{\sqrt{x-1}}$. Find the domain of f.

Solution: Notice that $f(x) = \dfrac{\sqrt{x}}{\sqrt{x-1}}$ is the quotient of $\sqrt{x}$ and $\sqrt{x-1}$, then

$$\begin{aligned} D(h) &= D(\sqrt{x}) \cap D(\sqrt{x-1}) \setminus \{x : x - 1 = 0\} \\ &= [0, \infty) \cap [1, \infty) \setminus \{1\} \\ &= [1, \infty) \setminus \{1\} = (1, \infty). \end{aligned}$$



□

The Composition of Functions

Definition .2: [Composition of two functions]
The composition of two functions f and g, written $f \circ g$, is defined by $(f \circ g)(x) = f(g(x))$ for all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Note 1: $D(f \circ g) = \{x : x \in D(g) \text{ and } g(x) \in D(f(g(x)))\} = D(g) \cap D(f(g(x)))$.

EXAMPLE 5. If $f(x) = x^2 + 4$ and $g(x) = \sqrt{x-4}$. Find the domain of the following: $f \circ g$, and $g \circ f$.

Solution:

1. We have $(f \circ g)(x) = f(g(x)) = f(\sqrt{x-4}) = (\sqrt{x-4})^2 + 4 = x - 4 + 4 = x$.
   $D(g) = [4, \infty)$ and $D(f(g(x))) = D(x) = \mathbb{R}$.
   Hence $D(f \circ g) = D(g(x)) \cap D(f(g(x))) = \mathbb{R} \cap [4, \infty) = [4, \infty)$.
2. We have $(g \circ f)(x) = g(f(x)) = g(x^2 + 4) = \sqrt{x^2 + 4 - 4} = \sqrt{x^2} = |x|$, $D(f) = \mathbb{R}$, and $D(g(f(x))) = D(|x|) = \mathbb{R}$.
   Hence $D(g \circ f) = D(f(x)) \cap D(g(f(x))) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

□

EXAMPLE 6. If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$. Find the domain of the following: $f\circ g$, and $g\circ f$.

Solution:

1. We have $(f\circ g)(x)=f(g(x))=f(\sqrt{1-x})=\sqrt{\sqrt{1-x}}=\sqrt[4]{1-x}$. $D(g)=(-\infty,1]$ and $D(\sqrt[4]{1-x})=(-\infty,1]$.
   Hence $D(f\circ g)=D(g(x))\cap D(f(g(x)))=(-\infty,1]$.

2. We have $(g\circ f)(x)=g(f(x))=g(\sqrt{x})=\sqrt{1-\sqrt{x}}$, and $D(f)=[0,\infty)$. To find $D(\sqrt{1-\sqrt{x}})$ we have two conditions $x\geq 0$ and $1-\sqrt{x}\geq 0$

$$
\begin{aligned}
1-\sqrt{x}\geq 0 &\Leftrightarrow 1\geq\sqrt{x} && \text{move } \sqrt{x} \text{ to the other side}\\
&\Leftrightarrow \sqrt{x}\leq 1 && \text{rewrite the inequality}\\
&\Leftrightarrow (\sqrt{x})^2\leq 1^2 && \text{square both sides}\\
&\Leftrightarrow x\leq 1 && \text{and since we have } 0\leq x,\\
&\Leftrightarrow 0\leq x\leq 1
\end{aligned}
$$

   Hence $D(\sqrt{1-\sqrt{x}})=[0,1]$
   Hence $D(g\circ f)=D(f(x))\cap D(g(f(x)))=[0,1]\cap[0,\infty)=[0,1]$.

□