King Abdulaziz University

0.4 Trigonometric Functions

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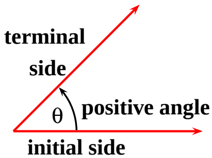
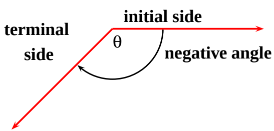
Trigonometric Functions

Angles:

An angle has three parts; an initial side , a terminal side, and a vertex. An angle θ is in standard position if its initial side is the positive x-axis so its vertex is at the origin. We say that θ is directed if a direction of rotation from its initial side to its terminal side is specified. We say that θ is a positive angle if this rotation is counterclockwise, otherwise the angle is a negative angle if it is clockwise. Angles can be measured in degrees($\circ$) or in radians (rad). The angle given by a complete revolution contains 360°, which is the same as 2π rad that is $2\pi = 360^\circ$. Hence π rad $= 180^\circ$ and

$$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ \quad 1^\circ = \frac{\pi}{180} \approx 0.017\text{rad}.$$





To convert degrees to radian: multiply by $\dfrac{\pi}{180}$.

To convert radian to degrees: multiply by $\dfrac{180}{\pi}$.

Example 1. Convert the following radians to degrees

1. $\dfrac{\pi}{4}$. 2. $\dfrac{\pi}{6}$. 3. $\dfrac{\pi}{3}$. 4. $\dfrac{4\pi}{3}$.

Solution:

1. $\dfrac{\pi}{4} = \dfrac{\cancel{\pi}}{4} \cdot \dfrac{180}{\cancel{\pi}} = 45^\circ$.
2. $\dfrac{\pi}{6} = \dfrac{\cancel{\pi}}{6} \cdot \dfrac{180}{\cancel{\pi}} = 30^\circ$.
3. $\dfrac{\pi}{3} = \dfrac{\cancel{\pi}}{3} \cdot \dfrac{180}{\cancel{\pi}} = 60^\circ$.
4. $\dfrac{4\pi}{3} = \dfrac{4\cancel{\pi}}{3} \cdot \dfrac{180}{\cancel{\pi}} = 240^\circ$.

□

Example 2. Convert the following degrees to radians

1. 40°. 2. 120°. 3. 15°. 4. 270°.

Solution:

1. $40^\circ = 40 \cdot \dfrac{\pi}{180} = \dfrac{2\pi}{9}$.
2. $120^\circ = 120 \cdot \dfrac{\pi}{180} = \dfrac{2\pi}{3}$.
3. $15^\circ = 15 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{12}$.
4. $270^\circ = 270 \cdot \dfrac{\pi}{180} = \dfrac{3\pi}{2}$.

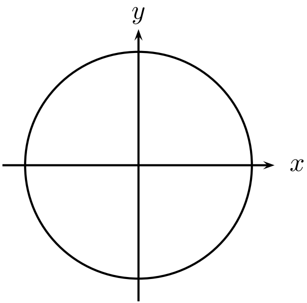
□

Definition .1: [Right Triangle Definition]

Let $0 < \theta < \dfrac{\pi}{2}$ using the right triangle we define the six trigonometric functions as follows

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$



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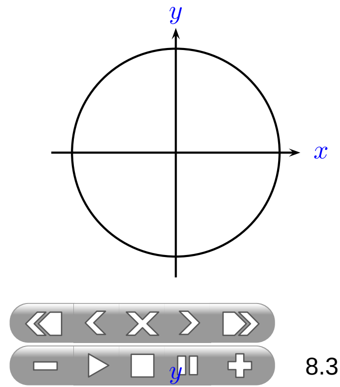
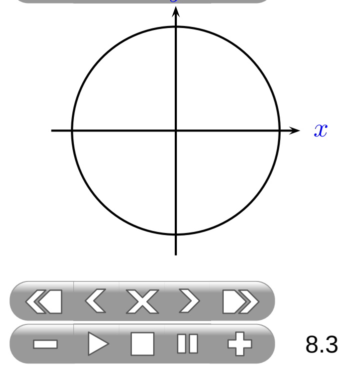
Definition .2: [Circular Definition]
Let θ be any angle using a circle of radius r we define the six trigonometric functions as follows

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$
$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$



Note that the coordinates of the point (x, y) on the circle $x^2 + y^2 = r^2$ can be expressed in term of θ and r as $x = r\cos\theta$ and $y = r\sin\theta$. Now, if $r = 1$ we see that $x = \cos\theta$ and $y = \sin\theta$. The table below lists some common values of sine and cosine.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0



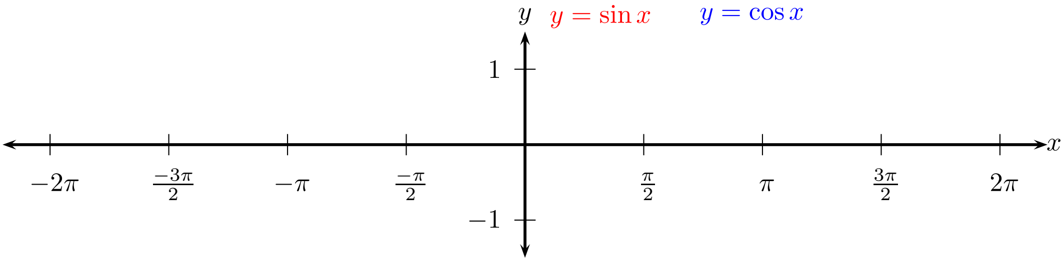
Definition .3: [Period and amplitude]

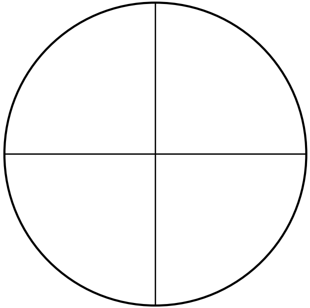
1. A function f is called periodic if there exists a positive constant p such that $f(x+p) = f(x)$ for any x in the domain of f. The smallest such number p is called the period of the function.
2. The amplitude of a periodic function f is defined to be one half the distance between its maximum value and its minimum value.

In the table below we list the important information about each the six trigonometric functions

Function	Domain	Range	Period	Amplitude	Symmetry
$\sin x$	$\mathbb{R}$	$[-1, 1]$	2π	1	$\sin(-x) = -\sin x$
$\cos x$	$\mathbb{R}$	$[-1, 1]$	2π	1	$\cos(-x) = \cos x$
$\tan x$	$\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}$	$\mathbb{R}$	π	none	$\tan(-x) = -\tan x$
$\cot x$	$\mathbb{R} \setminus \{k\pi\}$	$\mathbb{R}$	π	none	$\cot(-x) = -\cot x$
$\sec x$	$\mathbb{R} \setminus \{k\pi\}$	$(-\infty, -1] \cup [1, \infty)$	2π	none	$\sec(-x) = \sec x$
$\csc x$	$\mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}$	$(-\infty, -1] \cup [1, \infty)$	2π	none	$\csc(-x) = -\csc x$

Graphs of Trigonometric Functions





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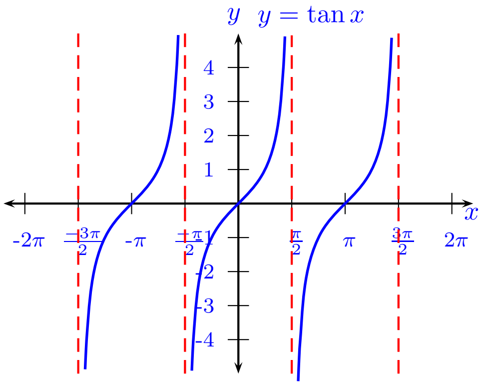
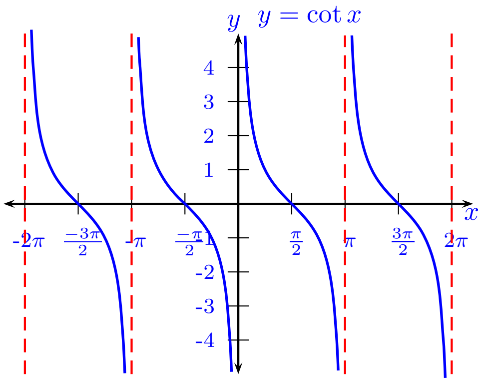
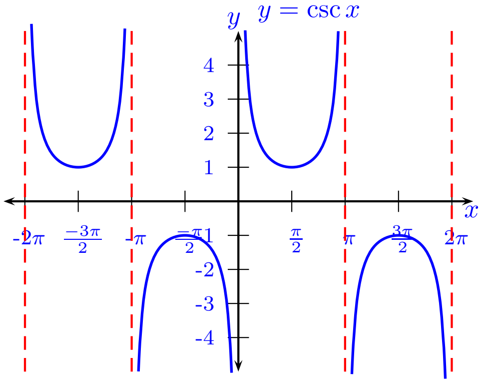
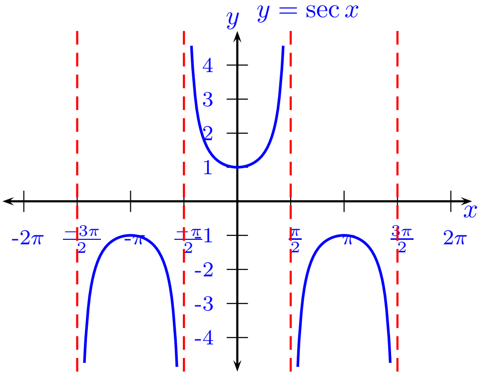

$y = \tan x$
y
x
4
3
2
1
-1
-2
-3
-4
-2π
$\frac{-3\pi}{2}$
-π
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


$y = \cot x$
y
x
4
3
2
1
-1
-2
-3
-4
-2π
$\frac{-3\pi}{2}$
-π
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


$y = \csc x$
y
x
4
3
2
1
-1
-2
-3
-4
-2π
$\frac{-3\pi}{2}$
-π
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π


$y = \sec x$
y
x
4
3
2
1
-1
-2
-3
-4
-2π
$\frac{-3\pi}{2}$
-π
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Trigonometric Identities:

Pythagorean Identities:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x$$

Reciprocal Identities:

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Sum-Difference of Two Angles:

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin y \sin x$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Quotient Identities:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Half-Angle Formulas:

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

Double-Angle Formulas:

$$\sin(2x) = 2 \sin x \cos y$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned}$$

EXAMPLE 3. Prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$.

Solution:

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) && \alpha-\beta=\alpha+(-\beta)\\
&= \sin\alpha\cos(-\beta)+\sin(-\beta)\cos\alpha && \cos(-x)=-\cos x \text{ and } \sin(-x)=-\sin x,\\
&= \sin\alpha\cos\beta+(-\sin\beta)\cos\alpha\\
&= \sin\alpha\cos\beta-\sin\beta\cos\alpha
\end{aligned}$$

□

EXAMPLE 4. Prove the identity $\sin(2x) = 2\sin x\cos x$.

Solution:

$$\begin{aligned}
\sin(2x) &= \sin(x+x) && 2x=x+x\\
&= \sin x\cos x+\sin x\cos x && \sin(x\pm y)=\sin x\cos y\pm\sin y\cos x,\\
&= 2\sin x\cos x
\end{aligned}$$

□

Example 5. Prove the identity $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ and $\cos^2 x = \dfrac{1+\cos(2x)}{2}$.

Solution:

$\cos(2x) = \cos(x+x)$	use $\cos(x+y) = \cos x \cos y - \sin y \sin x$
$= \cos x \cos x - \sin x \sin x$	$\cos x \cdot \cos x = \cos^2 x$ and $\sin x \cdot \sin x = \sin^2 x$,
$= \cos^2 x - \sin^2 x$	$\cos^2 x + \sin^2 x = 1 \Leftrightarrow \cos^2 x = 1 - \sin^2 x$
$\cos(2x) = 1 - \sin^2 x - \sin^2 x$	add and solve for $\sin^2 x$
$\cos(2x) = 1 - 2\sin^2 x$	move $-2\sin^2 x$ to the other side
$2\sin^2 x = 1 - \cos(2x)$	divide by 2
$\sin^2 x = \dfrac{1-\cos(2x)}{2}$	

$\cos(2x) = \cos^2 x - \sin^2 x$	$\cos^2 x + \sin^2 x = 1 \Leftrightarrow \sin^2 x = 1 - \cos^2 x$
$\cos(2x) = \cos^2 x - (1 - \cos^2 x)$	
$\cos(2x) = \cos^2 x - 1 + \cos^2 x$	add and solve for $\cos^2 x$
$\cos(2x) = 2\cos^2 x - 1$	add and solve for $\cos^2 x$
$2\cos^2 x = 1 + \cos(2x)$	divide by 2
$\cos^2 x = \dfrac{1+\cos(2x)}{2}$	

□

Example 6. Find $\cos\left(\frac{3\pi}{4}\right)$.

Solution:

$$\begin{aligned}
\cos\left(\frac{3\pi}{4}\right) &= \cos\left(\pi - \frac{\pi}{4}\right) && \frac{3\pi}{4} = \pi - \frac{\pi}{4} \\
&= \cos(\pi)\cos\left(\frac{3\pi}{4}\right) - \sin(\pi)\sin\left(\frac{3\pi}{4}\right) && \cos(x+y) = \cos x \cos y - \sin y \sin x, \\
&= (-1)\left(\frac{1}{\sqrt{2}}\right) - (0)\left(\frac{1}{\sqrt{2}}\right) && \cos(\pi) = -1, \sin(\pi) = 0. \\
&= -\frac{1}{\sqrt{2}}
\end{aligned}$$

□