King Abdulaziz University

0.2 Lines and Functions

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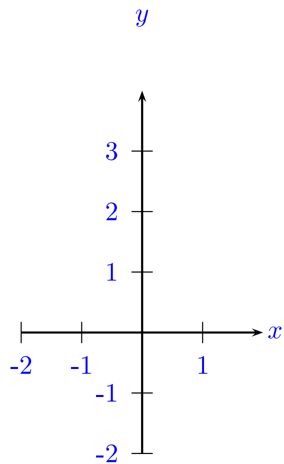
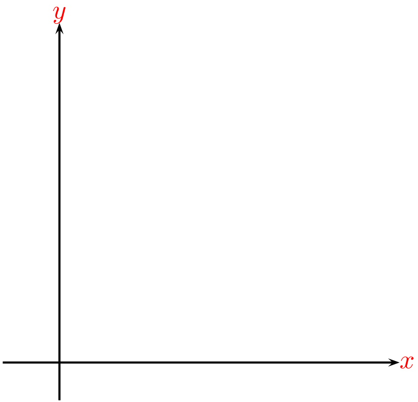
Lines

Lines rise or fall at a steady rate as we move a long them from left to right, unless, they are horizontal or vertical lines. Horizontal lines do not fall or rise at all, and on a vertical line we cannot move from left to right. The rate of rise or fall, the steepness, as we move from left to right along the line is called the *slope* of the line. We measure the slope in way that rising lines have a positive slopes, falling lines have negative slopes, horizontal lines have slope 0, and vertical lines have no slope at all.



slopes

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points on a nonvertical straight line l. We call $\Delta y = y_2 - y_1$, read " delta y", the rise from P_1 to P_2 and $\Delta x = x_2 - x_1$, read " delta x", the run from P_1 to P_2. Since the line is nonvertical, $\Delta x \neq 0$ and we defined the slope of the line to be $\Delta y / \Delta x$, the amount of rise per unit of run. We denoted the slope by m. Hence the slope of the line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $m = \dfrac{y_1 - y_2}{x_1 - x_2}$. For vertical line we can NOT use the formula $m = \Delta y / \Delta x$ because $\Delta x = 0$. We express this by saying vertical lines have no slope.



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Example 1. Find the slope of the line that passes through the points $(-1,5)$ and $(3,-3)$.

Solution: The slope of the line is

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-3-5}{3-(-1)} = \frac{-8}{4} = -2.$$

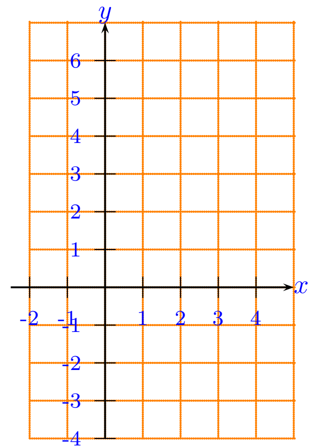
□

The points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are called *colinear* (means the points are on the same line) if $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{y_2 - y_3}{x_2 - x_3}$.

Example 2. Use slopes to determine whether the points $(-1,5)$, $(1,1)$, and $(3,-3)$ are colinear.

Solution: The slope of the line joining $(-1,5)$, $(1,1)$ is $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{1-5}{1-(-1)} = \dfrac{-4}{2} = -2$. The slope of the line joining $(1,1)$, $(3,-3)$ is $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-3-1}{3-1} = \dfrac{-4}{2} = -2$. Hence the points are colinear.

□



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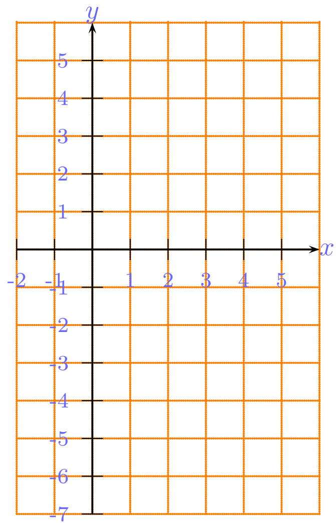
Equations of Lines

An equation of the line with slope m passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.(*Point-Slope form*) The equation $y = mx + b$ is the *Slope-Intercept equation* of the line of slope m and y-intercept $b(0, b)$.



Example 3. Find an equation of the line that has a slope of 2 and passes through the point $(2, -1)$.

Solution:

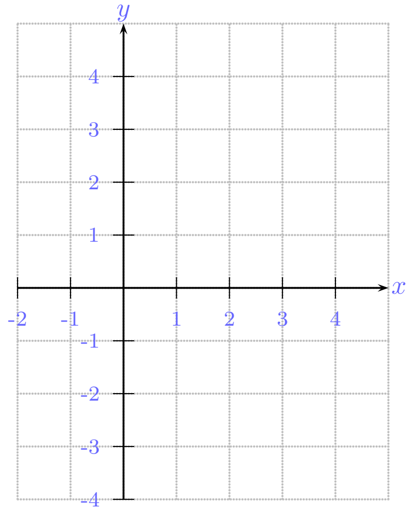
$y - y_1 = m(x - x_1)$	Point-Slope form
$y - (-1) = 2(x - 2)$	Substitute -1 for y_1, 2 for x_1, and 2 for m.
$y + 1 = 2x - 4$	Simplify
$y = 2x - 5$	Solve for y.

□

8.3

Summary of Equations of Lines

General form:	$Ax + By + C = 0$
Vertical line:	$x = a$
Horizontal line:	$y = b$
Point-Slope form:	$y - y_1 = m(x - x_1)$
Slope-Intercept form:	$y = mx + b$



EXAMPLE 4. Sketch the graph of $x = 2$, $y = -1$, and $2y + 3x - 6 = 0$.

Solution:

$x = 2$	is a vertical line.
$y = -1$	is a horizontal line.
$2y + 3x - 6 = 0$	Isolate y term.
$2y = -3x + 6$	Solve for y.
$y = -3/2x + 3$	Point-Slope form

8.3

Parallel Lines

Two distinct nonvertical line are parallel if they have the same slope. That is two lines with slopes m_1 and m_2 are parallel if $m_1 = m_2$. Any two vertical lines are parallel.

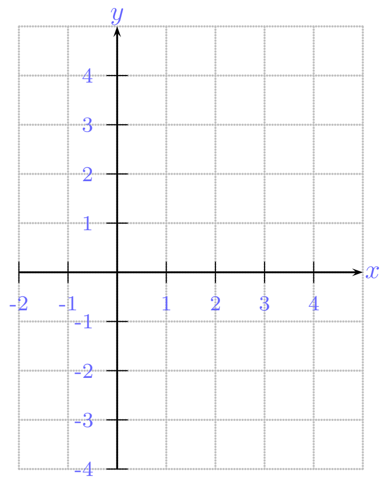
Example 5. Find an equation for the line that is parallel to $2x - 3y = 5$ and passes through $(2, -1)$.

Solution: we find the slope of the line $2x - 3y = 5$

$-3y = -2x + 5$	Isolate y term.
$y = -2/-3x + 5/-3$	Solve for y.
$y = 2/3x - 5/3$	Point-Slope form

Now, use the $m = 2/3$ and the point $(2, -1)$.

$y - y_1 = m(x - x_1)$	Point-Slope form
$y - (-1) = 2/3(x - 2)$	Substitute
$y + 1 = 2/3x - 4/3$	Simplify
$3y + 3 = 2x - 4$	Simplify
$2x - 3y - 7 = 0$	



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Perpendicular Lines

Two nonvertical line are perpendicular if the product of their slopes equal to -1. That is two lines with slopes m_1 and m_2 are perpendicular if $m_1 = -1/m_2$. Any vertical and any horizontal lines are perpendicular.

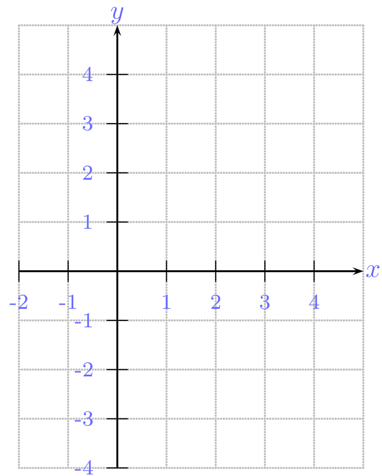
EXAMPLE 6. Find an equation for the line that is perpendicular to $2x - 3y = 5$ and passes through $(2, -1)$.

Solution: we find the slope of the line $2x - 3y = 5$

$$-3y = -2x + 5 \quad \text{Isolate } y \text{ term.}$$
$$y = -2/-3x + 5/-3 \quad \text{Solve for } y.$$
$$y = 2/3x - 5/3 \quad \text{Point-Slope form}$$

Now, use the $m = -3/2$ and the point $(2, -1)$.

$$y - y_1 = m(x - x_1) \quad \text{Point-Slope form}$$
$$y - (-1) = -3/2(x - 2) \quad \text{Substitute}$$
$$y + 1 = -3/2x + 3 \quad \text{Simplify}$$
$$2y + 2 = -3x + 6 \quad \text{Simplify}$$
$$3x + 2y - 4 = 0$$



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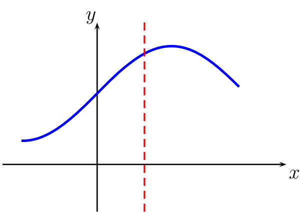
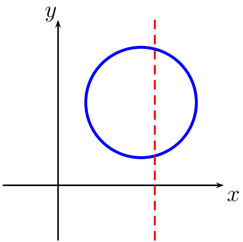
Functions

Definition .1: [Function]

A function f from a set A to a set B is a rule which assigns to each element x of the set A a unique element y of the set B. We write $y = f(x)$. This is read " y equals f of x. " A the set of possible inputs or the set of values of x is called the domain of the function f. The range of a function f is the set $\{f(x) : x \in A\}$. In other words, the range is the set of possible outputs. The range is a subset of B. x is called an independent variable, y is called a dependent variable. By the graph of a function f, we mean the graph of the equation $y = f(x)$. Notice that not every curve is a graph of a function, because for a function for a given value of x we should have only one value of y. We use the **vertical line test** to determine whether the the curve is a graph of a function or not. **vertical line test**: if any vertical line intersects the graph in more than one point, the curve is not the graph of a function



The graph represents a function.



This graph does not represent a function.

Types of Functions

Definition .2: [Polynomial Functions]

A function the form

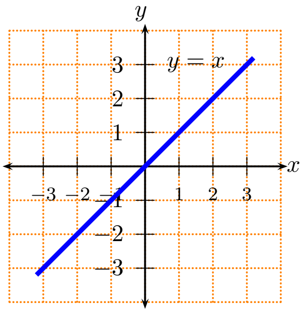
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \quad a_n \neq 0,$$

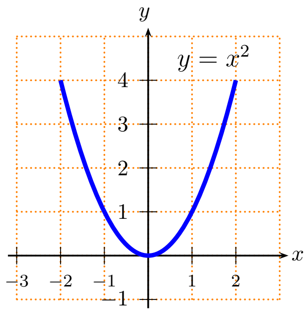
is called a polynomial where n is nonnegative integers and $a_n, a_{n-1}, \cdots, a_2, a_1, a_0$ are constants called the coefficients of the polynomial. The domain of the polynomial is the real numbers $\mathbb{R}$ and the degree of the polynomial is n.

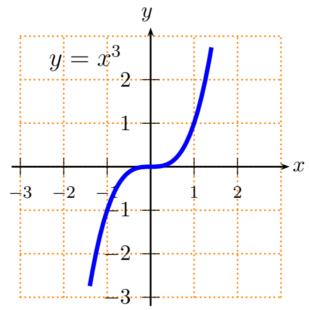
Note 1: The domain of any polynomial is the set of real numbers $\mathbb{R} = (-\infty, \infty)$ and the range is subset of $\mathbb{R}$. Some of the polynomials have spacial names and they are known by those name. The following simpler forms are often used.

Zeroth degree:	$f(x) = a$	Constant function.
First degree:	$f(x) = ax + b$	Linear function.
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function.
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function.

The Figures below show the graphs of the basic polynomials. You should be able to recognize these graphs and be able to sketch them.







Example 7. Find the domain of $f(x) = 2x^2 - x + 1$, and $g(x) = 2/3x^7 - 8/\sqrt{2}x^3 - 7$.

Solution: Since $f(x) = 2x^2 - x + 1$, and $g(x) = 2/3x^7 - 8/\sqrt{2}x^3 - 7$ are polynomials then

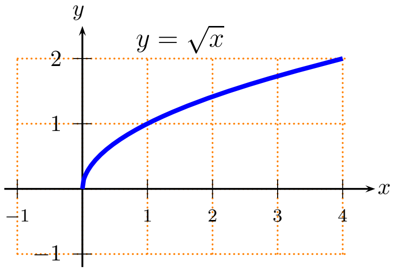
$$D(f) = \mathbb{R} = D(g).$$

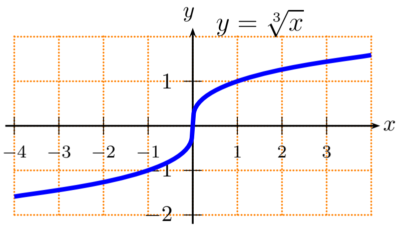
□

Definition .3: [Root Functions]

1. A function is called an **even root function**, if $f(x) = \sqrt[n]{p(x)}$ where $p(x)$ is polynomial and n is an even positive integers. The domain of the even root function is
$$D(f) = \{x : p(x) \geq 0\}.$$
2. A function is called an **odd root function**, if $f(x) = \sqrt[n]{p(x)}$ where $p(x)$ is polynomial and $n \geq 3$ is an odd positive integers. The domain of the odd root function is the real numbers
$$D(f) = \mathbb{R}.$$

Below is the graphs of the square root function as an example of even root function and the cubic root function as an example of an odd root function.



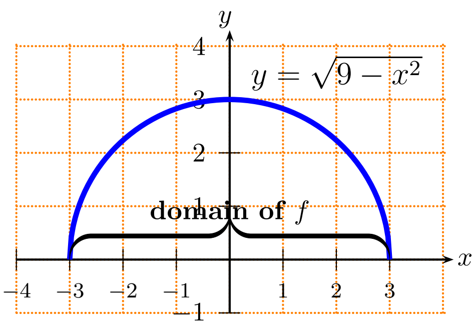


EXAMPLE 8. Find the domain of $f(x) = \sqrt{9 - x^2}$.

Solution: We have an even root function, the domain consists of all numbers x such that, $9 - x^2 \geq 0$.

$$
\begin{aligned}
9 - x^2 \geq 0 &\Leftrightarrow 9 \geq x^2 && \text{move } x^2 \text{ to the other side} \\
&\Leftrightarrow x^2 \leq 9 && \text{rewrite the inequality} \\
&\Leftrightarrow \sqrt{x^2} \leq 3 && \text{take the square root} \\
&\Leftrightarrow |x| \leq 3 && \sqrt{x^2} = |x| \text{ use properties of} \\
&\Leftrightarrow -3 \leq x \leq 3 && \text{absolute value inequality}
\end{aligned}
$$

Hence $D(f) = [-3, 3]$.



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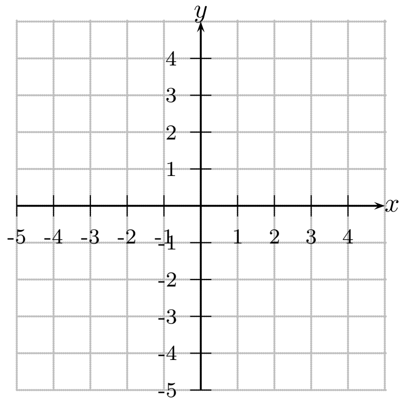
EXAMPLE 9. Find the domain of $g(x) = \sqrt[3]{1-x}$

Solution:

We have an odd root function, the domain consists of all real numbers. Hence $D(g) = \mathbb{R}$.



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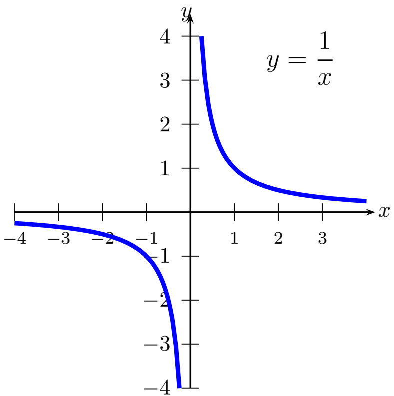
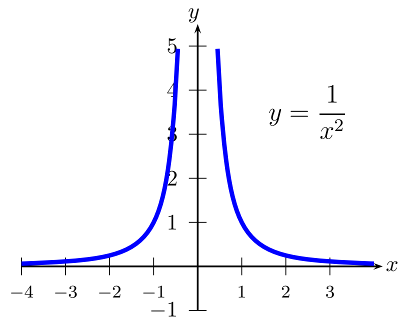
Definition .4: [Rational Functions]

A function is called rational function, if $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomial functions. The domain of the rational function is

$$D(f) = \mathbb{R} \setminus \{x \mid q(x) = 0\} = \mathbb{R} \setminus \{ \text{ zeroes of } q(x)\}.$$

Below is the graphs of some rational functions,





Note 2: For some reason, students think that any function, which has variables in a denominator, is a rational function. This is not true. For example, the function $f(x) = \dfrac{x-1}{\sqrt{x+4}}$ is not a rational function beacons its denominator is not a polynomial. So a rational function has to have a polynomial in its numerator and its denominator.

EXAMPLE 10. Find the domain of the following function $f(x) = \dfrac{x}{8-2x}$

Solution: We have a rational function, find the zeros of the bottom, $8-2x=0$, $x=4$. Hence $D(f) = \mathbb{R} \setminus \{4\} = (-\infty, 4) \cup (4, \infty)$.

□

EXAMPLE 11. Find the domain of $f(x) = \dfrac{x^2+1}{12x^2-60x+72}$

Solution: Set $12x^2-60x+72=0$,

$$\begin{aligned} 12(x^2-5x+6) &= 0 \\ 12(x-3)(x-2) &= 0 \quad \text{factoring} \\ x-3=0 \text{ or } x-2 &= 0 \quad 12 \neq 0 \\ x=3 \text{ or } x &= 2 \end{aligned}$$

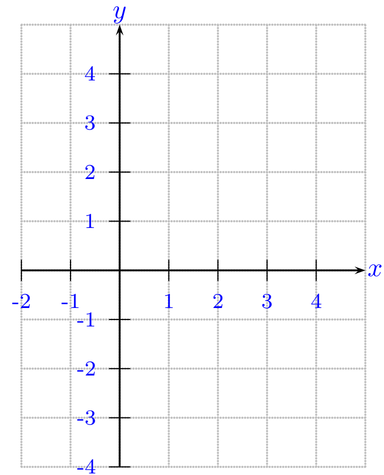
Hence $D(f) = \mathbb{R} \setminus \{2,3\} = (-\infty, 2) \cup (2,3) \cup (3, \infty)$.

□

Definition .5: [x-intercepts and y-intercept]

Let $y = f(x)$ be a function.

1. The values of x for which $f(x) = 0$ are the x-coordinate of the points where the graph of f intersects the x-axis. These values are called ***zeros*** of f, the ***roots*** of $f(x) = 0$, or the ***x-intercepts*** of $y = f(x)$.
2. The point $(0, f(0))$ is called the ***y-intercept*** which is where the graph of f intersects the y-axis.



Example 12. Find all intercepts of the graph of $f(x) = x^2 - 5x + 6$

Solution: x-intercepts occurs where

$$x^2 - 5x + 6 = 0$$
$$(x-3)(x-2) = 0 \quad \text{factoring}$$
$$x - 3 = 0 \text{ or } x - 2 = 0 \quad 12 \neq 0$$
$$x = 3 \text{ or } x = 2$$

y-intercept at $f(0) = (0)^2 - 5(0) + 6 = 6$. □

8.3

EXAMPLE 13. Find all zeros of the graph of
$f(x) = 2x^2 + 4x - 1$

Solution: We use the quadratic formula to find the zeros of $f(x) = 2x^2 + 4x - 1$. Notice that $a = 2$, $b = 4$, and $c = -1$. Hence

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

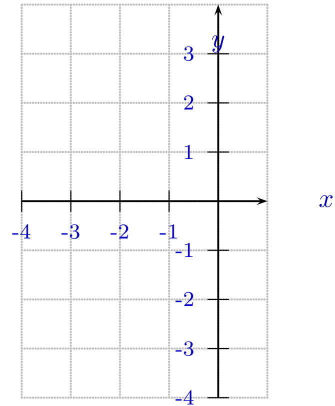
$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(-1)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 + 8}}{4}$$

$$x = \frac{-4 \pm \sqrt{24}}{4}$$

$$x = \frac{-4 \pm \sqrt{4}\sqrt{6}}{4}$$

$$x = \frac{-4 \pm 2\sqrt{6}}{4} = \frac{-2 \pm \sqrt{6}}{2}$$



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