NORMED AND BANACH SPACES

1. Normed and Banach Spaces

Definition 1.1:[Normed Space]

Let X be a linear space over the field $\mathbb{F}$. A mapping $\| \,.\, \|: X \longrightarrow \mathbb{R}$ is said to be a norm on x if it satisfies the following conditions

(1) $\| x \| \geq 0 \quad \forall x \in X$

(2) $\| x \| = 0 \Leftrightarrow x = 0$

(3) $\| \alpha x \| = |\alpha| \, \| x \|, \quad \forall x \in X$ (Homogenity of norm)

(4) $\| x + y \| \leq \| x \| + \| y \| \quad \forall x, y \in X$ (Triangle Inequality)

Example 1: The linear spaces $\mathbb{R}, \mathbb{C}$ are both normed spaces with norm given by $\|x\| = |x|$.

Note 1: A norm $\|.\|$ on X defines a metric d on X given by $d(x,y) = \|x - y\|$.

Note that $d(x-y, 0) = \|(x-y) - 0\| = \|x - y\| = d(x,y)$ and $d(ax, 0) = \|ax\| = |a| \, \|x\| = |a| d(x,0)$.

Definition 1.2:[]

Let X be a normed space over the field $\mathbb{F}$.

(1) The open ball center at $x_0 \in X$ with radius $r > 0$ is the set $B_r(x_0) = \{x \in X : \|x - x_0\| < r\}$.

(2) A sequence $\{x_n\} \subset X$ is called ***convergent*** if $\exists x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

(3) A sequence $\{x_n\} \subset X$ is called ***Cauchy*** if $\forall \varepsilon > 0, \quad \exists N \in \mathbb{N}$ such that if $n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon$.

(4) The normed space X is ***complete*** if every Cauchy sequence in X is convergent in X.

(5) The normed space X is ***Banach space*** if X is complete.

Note 2: Let X be a normed space. Let $x_0 \in X$, and $r > 0$, then

$$\begin{aligned} B_r(x_0) &= \{x \in X : \|x - x_0\| < r\} \text{ let } y = x - x_0 \Leftrightarrow x = x_0 + y \\ &= \{x_0 + y : \|y\| < r\} \\ &= x_0 + \{y : \|y\| < r\} \\ &= x_0 + B_r(0) \end{aligned}$$

Thus the open ball center at any point in X is the translate of the ball center at 0 with the same radius. Also,

$$\begin{aligned} B_r(0) =& \{x \in X : \|x\| < r\} \\ =& \{x : \left\|\frac{x}{r}\right\| < 1\} \qquad \text{let } y = \frac{x}{r} \Leftrightarrow x = ry \\ =& \{ry : \|y\| < 1\} \\ =& r\{y : \|y\| < 1\} \\ =& rB_1(0) \end{aligned}$$

Hence $B_r(x_0) = x_0 + B_r(0) = x_0 + rB_1(0)$. Thus in any normed space we can consider the unit open ball.

Lemma 1: Let X be a normed space over $\mathbb{F}$. Let $\{x_n\}, \{y_n\} \subset X$ such that $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} y_n = y \in X$, and let $\{a_n\} \subset \mathbb{F}$ such that $\lim_{n\to\infty} a_n = a \in \mathbb{F}$. Then

(a) $|\,\|x\| - \|y\|\,| \leq \|x - y\|, \quad \forall x, y \in X.$

(b) $\lim_{n\to\infty} (x_n + y_n) = x + y.$

(c) $\lim_{n\to\infty} a_n x_n = ax.$

Proof:

(a)

$$\begin{aligned} \|x\| &= \|x - y + y\| \\ &\leq \|x - y\| + \|y\| \\ \|x\| - \|y\| &\leq \|x - y\| \\ \|y\| &= \|y - x + x\| \\ &\leq \|y - x\| + \|x\| \\ \|y\| - \|x\| &\leq \|y - x\| = \|x - y\| \\ -(\|x\| - \|y\|) &\leq \|x - y\| \\ \text{Thus } \|x\| - \|y\| &\geq -\|x - y\| \\ \text{Hence } -\|x - y\| &\leq \|x\| - \|y\| \leq \|x - y\| \end{aligned}$$

(b)

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &= \|x_n - x + y_n - y\| \\ &\leq \|x - n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

(c) Since $\lim_{n\to\infty} a_n = a$, then $\exists M > 0 \ni |a_n| \le M \quad \forall n \ge 1$.

$$\begin{aligned}\|a_n x_n - ax\| &= \|a_n x_n - a_n x + a_n x - ax\| \\ &\le \|a_n x_n - a_n x\| + \|a_n x - ax\| \\ &\le |a_n|\,\|x_n - x\| + |a_n - a|\,\|x\| \\ &\le M\,\|x_n - x\| + |a_n - a|\,\|x\| \quad \to 0 \text{ as } n \to \infty.\end{aligned}$$

Definition 1.3:[]

Let X be a normed space over the field $\mathbb{F}$. Let $\{x_n\} \subset X$ be a sequence and for each $n \ge 1$, let $s_n = \sum_{k=1}^{n} x_k$. The sequence $\{s_n\}$ is called the sequence of partial sums. The sequence $\{x_n\}$ is called ***summable*** to $s \in X$ if $\{s_n\}$ converges. Thus $\{x_n\}$ is called summable if $\lim_{n\to\infty} \|s_n - s\| = 0$. The sequence $\{x_n\}$ is called ***absolutely summable*** if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem 1.1: []

A normed space X is a Banach space iff every absolutely summable sequence in X is summable in X.

Proof: $(\Rightarrow)$ Suppose that X is a Banach space. Let $\{x_n\}$ be an absolutely summable sequence in X. Then $\sum_{n=1}^{\infty} \|x_n\| = M < \infty$. Hence for each $\varepsilon > 0$, $\exists \; N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \|x_n\| < \varepsilon$.
Now,

$$\begin{aligned}\text{if } n \ge m > N \Rightarrow \|s_n - s_m\| &= \left\| \sum_{k=m+1}^{n} x_k \right\| \\ &\le \sum_{k=m+1}^{n} \|x_k\| \\ &\le \sum_{k=N}^{\infty} \|x_k\| < \varepsilon.\end{aligned}$$

Thus $\{s_n\}$ is a Cauchy sequence in X, hence $\{s_n\}$ is convergent since X is Banach space. Therefore $\{x_n\}$ is summable.
$(\Leftarrow)$ Suppose each absolutely summable sequence in X is summable in X. Let $\{x_n\} \subset X$ be a Cauchy sequence in X. Now, since $\{x_n\}$ is Cauchy, $\exists \; n_1 \in \mathbb{N}$ such that if $n, m \ge n_1 \Rightarrow \|x_n - x_m\| < \frac{1}{2}$. Also, $\exists \; n_2' \in \mathbb{N}$ such that if $n, m \ge n_2' \Rightarrow \|x_n - x_m\| < \frac{1}{2^2}$, and let $n_2 > \max\{n_1, n_2'\}$. Now, $n_2 > n_1$ and if $n, m \ge n_2 \Rightarrow \|x_n - x_m\| < \frac{1}{2^2}$. Hence $n_1, n_2 \ge n_2 \Rightarrow \|x_{n_2} - x_{n_1}\| < \frac{1}{2^2}$. Continuing this way, we have for each $k \ge 2 \quad \exists n_{k+1} > n_k$ such that $\left\|x_{n_{k+1}} - x_{n_k}\right\| < \frac{1}{2^k}$. Now, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Set $y_0 = x_{n_1}$ and $y_k = x_{n_{k+1}} - x_{n_k} \quad \forall,\ k \ge 1$. Note that $\|y_k\| = \left\|x_{n_{k+1}} - x_{n_k}\right\| < \frac{1}{2^k}$ and $\sum_{i=1}^{k} y_i = x_{n_{k+1}}$.
Now, $\sum_{k=0}^{\infty} \|y_k\| = \|y_0\| + \sum_{k=1}^{\infty} \|y_k\| \le \|y_0\| + \sum_{k=1}^{\infty} \frac{1}{2^k} = \|y_0\| + 1 < \infty$. Thus $\{y_k\}$ is absolutely summable and hence it summable by assumption. Hence $\sum_{k=0}^{\infty} y_k = x \in X$. Now $\lim_{k\to\infty} x_{n_{k+1}} = \lim_{k\to\infty} \sum_{i=1}^{k} y_i = \sum_{i=0}^{\infty} y_i = x$. Thus $\lim_{k\to\infty} x_k = x \in X$. Thus, the Cauchy sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ convergent to x. Therefore $\lim_{n\to\infty} x_n = x \in X$. Hence X is a Banach space.

Example 2: Let p be a real number such that $1 \leq p < \infty$. l_p is the space of all sequence $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{F}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty \quad (x = \{x_n\}_{n=1}^{\infty}$ converges).

$l_p = \{x = \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} |x_n|^p < \infty, \quad x_n, \in \mathbb{F}, \forall n \in \mathbb{N}\}$ with the norm is given by $\|x\|_p = \sqrt[p]{\sum_{n=0}^{\infty} |x_n|^p}, \quad x = \{x_n\} \in l_p$ is a Banach space. To see that we will prove the tringle inequality and the completness. Let $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ in l_p, then using Minkowski Inequality. we have

$$\begin{aligned}\|x+y\|_p &= \sqrt[p]{\sum_{n=0}^{\infty} |x_n + y_n|^p} \\ &\leq \sqrt[p]{\sum_{n=0}^{\infty} |x_n|^p} + \sqrt[p]{\sum_{n=0}^{\infty} |y_n|^p} \\ &= \|x\|_p + \|y\|_p .\end{aligned}$$

Let $\{x_k\}$, where $x_k = \{x_n^{(k)}\}$, be a Cauchy sequence in l_p such that $\sum_{n=1}^{\infty} |x_n^{(k)}|^p < \infty, \quad k \geq 1$. Now, for each $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$ if $n, m \geq N \Rightarrow \|x_n - x_m\|_p = \sqrt[p]{\sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p} < \frac{\varepsilon}{2}$. Now, $|x_i^{(n)} - x_i^{(m)}| = \sqrt[p]{|x_i^{(n)} - x_i^{(m)}|^p} \leq \|x_n - x_m\|_p < \frac{\varepsilon}{2}$. Thus for each fixed $i (1 \leq i < \infty)$, the sequence $\{x_i^{(n)}\}$ is Cauchy in $\mathbb{F}$ which is complete. Hence the sequence $\{x_i^{(n)}\}$ is convergent. Hence for each $1 \leq i < \infty, \quad \lim_{n \to \infty} x_i^{(n)} = x_i \in \mathbb{F}$. Now, let $x = \{x_i\}$. Now, if $n, m \geq N \Rightarrow \|x_n - x_m\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < (\frac{\varepsilon}{2})^p$. Letting $m \to \infty$, we obtain, $\|x_n - x\|_p \leq \frac{\varepsilon}{2} < \varepsilon$. Hence $x_n - x = \{x_i^{(n)} - x_i\} \in l_p$ Since $x_n, x_n - x \in l_p \Rightarrow x = x_n + (x - x_n) \in l_p$. Thus $\{x_k\}$, convergent to $x \in l_p$. Hence l_p is a Banach space.

Example 3: Let a, b be two real numbers such that $a < b$. Consider $C([a,b])$ is the space of all continuous functions f over $[a,b]$,

$C([a,b]) = \{f : [a,b] \to \mathbb{F} \mid f$ is continuous on $[a,b]\}$, with norm $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$. Then $(C([a,b]), \|.\|_\infty)$ is a Banach space. It is easy task to check that $\|.\|_\infty$ is a norm. We will prove the completeness of $C([a,b])$. Let $\{f_n\}$ be a Cauchy sequence in $C([a,b])$. Then, for each $\varepsilon > 0$, $\exists N \in \mathbb{N} \ni$ if $n, m \geq N \Rightarrow \|f_n - f_m\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. Thus, for a fixed $x_0 \in [a,b]$, we have if $n, m \geq N \Rightarrow |f_n(x_0) - f_m(x_0)| \leq \|f_n - f_m\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. Hence the sequence $\{f_n(x_0)\}$ is a Cauchy sequence in $\mathbb{F}$, since $\mathbb{F}$ is Banach space, then this sequence converges. Let $f(x_0) = \lim_{n \to \infty} f_n(x_0)$. Now, for each $x \in [a,b]$, let $f(x) = \lim_{n \to \infty} f_n(x)$. Now, we have if $n, m \geq N \Rightarrow |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty = \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall x \in [a,b]$. Letting $m \to \infty$, we obtain, $\|f_n - f\|_\infty \leq \frac{\varepsilon}{2} < \varepsilon$ and since f_n is continuous and convergence is uniformly, then $f \in C([a,b])$. Therefore $C([a,b])$ is Banach space.

Example 4: Consider the space $C([-1,1])$ equipped with the norm $\|f\|_1 = \int_{-1}^{1} |f(x)|\,dx$. We will show that $(C([-1,1]), \|.\|_1)$ is not Banach space. Let $\{f_n\}$ be the sequence in $C([-1,1])$, where $f_n(x) = \begin{cases} 1, & \text{if } -1 \le x \le 0; \\ -nx+1, & \text{if } 0 < x < \frac{1}{n}; \\ 0, & \text{if } \frac{1}{n} < x \le 1. \end{cases}$

Below the graphs of f_n and $f_n - f_m$ for $m > n$.

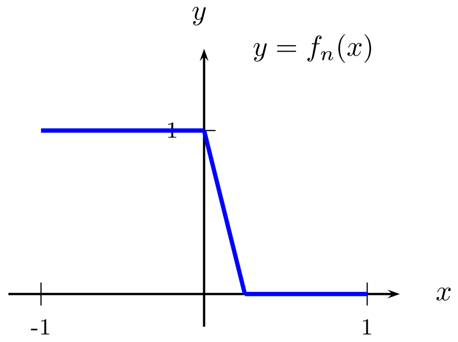

FIGURE 1

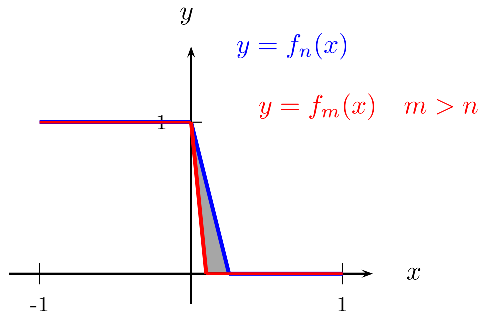

FIGURE 2

Now, Since

$$\|f_n - f_m\|_1 = \frac{1}{2n} - \frac{1}{2m} \to 0 \text{ as } n,m \to \infty,$$

hence $\{f_n\}$ is Cauchy sequence in $C([-1,1])$. Suppose there is $f \in C([-1,1])$ such that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. Hence

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|f_n - f\|_1 \\ &= \lim_{n\to\infty} \int_{-1}^{1} |f_n(x) - f(x)|\,dx \\ &= \lim_{n\to\infty} \left[\int_{-1}^{0} |1 - f(x)|\,dx + \int_{0}^{1/n} |f_n(x) - f(x)|\,dx + \int_{1/n}^{1} |f(x)|\,dx \right] \end{aligned}$$

Hence $\int_{-1}^{0} |1-f(x)|\,dx = 0 \Rightarrow |1-f(x)| = 0 \quad \forall x \in [-1,0] \Rightarrow f(x) = 1 \quad \forall x \in [-1,0]$.

Also, $\lim_{n\to\infty} \int_{1/n}^{1} |f(x)|\,dx = 0 \Rightarrow f(x) = 0 \quad \forall x \in (0,1]$. Therefore $f(x) = \begin{cases} 1, & \text{if } -1 \le x \le 0; \\ 0, & \text{if } 0 < x \le 1. \end{cases} \quad \notin C([-1,1])$.

Hence $(C([-1,1]), \|.\|_1)$ is not Banach space.

Example 5: Consider the space $C^1([0,1]) = \{f : [0,1] \to \mathbb{F} : f' \in C([0,1])\}$ equipped with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$.

We will show that $(C^1([0,1]), \|.\|_\infty)$ is not Banach space. Let $\{f_n\}$ be the sequence in $C^1([0,1])$, where $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ Below the graphs of some of f_n and some of $f_n - f_m$ for $m > n$.

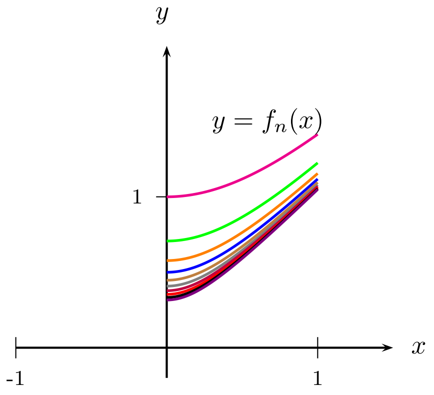

FIGURE 3

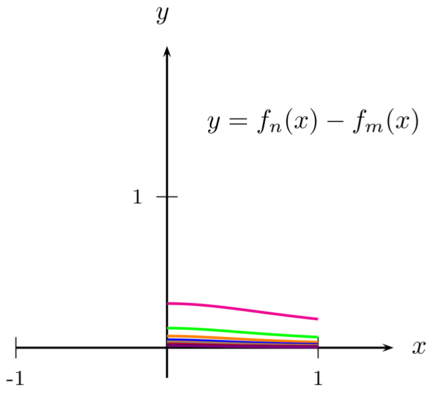

FIGURE 4

Now, Since

$$\|f_n - f_m\|_\infty = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}} \longrightarrow 0 \text{ as } n,m \longrightarrow \infty,$$

hence $\{f_n\}$ is Cauchy sequence in $C^1([0,1])$. Now, $f(x) = \lim_{n\to\infty} f_n(x) = \sqrt{x^2} = |x|$, hence $\{f_n(x)\}$ converges pointwise to $f(x) = |x|$.

$$\begin{aligned}
\|f_n - f\|_\infty &= \sup_{x\in[0,1]} |f_n(x) - f(x)| \\
&= \sup_{x\in[0,1]} \left| \sqrt{x^2 + \frac{1}{n}} - |x| \right| \\
&= \frac{1}{\sqrt{n}} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Hence the convergence is uniformly on $[0,1]$, but $f(x) = |x|$ is not differentiable at $x = 0$. Thus $f(x) = |x| \notin C^1([0,1])$. Hence $(C^1([0,1]).,\|.\|_\infty)$ is not Banach space.

1.1. **Subspaces and Quotient Spaces.**

Definition 1.4:[Closed subspace]

Let X be a normed space and Y be a linear subspace of X. We say that Y is a closed subspace if Y is a closed subset of X under the norm topology.

Example 6: Consider $l_\infty = \{x = \{x_n\}_{n=1}^{\infty} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ with the norm $\|\{x_n\}_{n=1}^{\infty}\| = \sup_{n\in\mathbb{N}} |x_n|$. Now, the space $c = \{x = \{x_n\}_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n \in \mathbb{F}, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ is a closed subspace of l_∞.

Also, $c_0 = \{x = \{x_n\}_{n=1}^{\infty} \mid \lim_{n\to\infty} x_n = 0, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ is a closed subspace of c.

Theorem 1.2: []

Let Y be a subspace of a Banach space X. Then Y is a Banach subspace (complete) iff Y is closed.

Proof: $(\Rightarrow)$ Suppose that Y is a Banach subspace. Let $x \in \overline{Y}$. Then, for each $n \geq 1$ there is $x_n \in (B_{1/n}(x) \cap (Y \setminus \{x\}))$. Now, $\{x_n\} \subset Y$ such that $\|x_n - x\| < \dfrac{1}{n} \quad \forall n \geq 1$. Thus $\lim_{n\to\infty} x_n = x$. Hence $\{x_n\}$ is a Cauchy sequence in Y and must be converge in Y because Y is complete. Thus $x \in Y$ and hence Y is closed.

$(\Leftarrow)$ Suppose that Y is closed. Let $\{x_n\}$ be a Cauchy sequence in Y and hence in X. Since X is a Banach space then there is $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ is a sequence in Y and Y is closed, then $x \in \overline{Y} = Y$. Thus $\{x_n\}$ is convergent in Y. Thus Y is a Banach subspace.

Theorem 1.3: [Quotient Space]

Let X be a normed space over $\mathbb{F}$ and let M be a closed subspace of X. Define $\|.\|_q : \frac{X}{M} \to \mathbb{R}$ by $\|x+M\|_q = \inf_{m\in M} \|x+m\|$. Then $(\frac{X}{M}, \|.\|_q)$ is a normed space. Moreover, if X is a Banach space, then $\frac{X}{M}$ is a Banach space.

Proof: We know that the quotient space $\frac{X}{M} = \{x+M : x \in X\}$ is a liner space. We will show $\|.\|_q$ is a norm.

1. Since $\|x+m\| \geq 0 \quad \forall x \in X$ and $\forall m \in M$, then $\|x+M\|_q \geq 0$.
2. Note that if $x+M = M \Rightarrow \|x+M\|_q = \|0+M\|_q = \|0\| = 0$. Now, let $\|x+M\|_q = 0$ for some $x \in X$. Then, for each $n \geq 1, \exists m_n \in M \ni \|x+m_n\| < \|x+M\| + \frac{1}{n} = \frac{1}{n}$. Hence $\lim_{n\to\infty} \|x+m_n\| = 0 \Rightarrow -m_n \to x$ as $n \to \infty$. But, since M is closed, then $x \in M \Rightarrow x+M = M$. Thus $\|x+M\|_q = 0 \Leftrightarrow x+M = M$.
3. For $x \in X$ and $\alpha \in \mathbb{F}$, $\alpha \neq 0$, we have

$$\begin{aligned}
\|\alpha(x+M)\|_q &= \|\alpha x+M\|_q \\
&= \inf_{m\in M} \|\alpha x+m\| \qquad \text{let } m' = \frac{m}{\alpha} \\
&= \inf_{m'\in M} \|\alpha x+\alpha m'\| \\
&= \inf_{m'\in M} |\alpha| \|x+m'\| \\
&= |\alpha| \inf_{m'\in M} \|x+m'\| \\
&= |\alpha| \|x+M\|_q
\end{aligned}$$

4. For $x, y \in X$, we have

$$
\begin{aligned}
\|(x+M)+(y+M)\|_q &= \|(x+y)+M\|_q \\
&= \inf_{m\in M} \|(x+y)+m\| \qquad \text{let } m = m_1+m_2, \quad m_1, m_2 \in M \\
&= \inf_{m_1,m_2\in M} \|(x+m_1)+(y+m_2)\| \\
&\le \inf_{m_1,m_2\in M} \{\|x+m_1\|+\|y+m_2\|\} \\
&\le \inf_{m_1\in M} \|x+m_1\| + \inf_{m_2\in M} \|y+m_2\| \\
&= \|x+M\|_q + \|y+M\|_q
\end{aligned}
$$

Suppose that X is a Banach space. Let $\{x_n+M\}$ be a Cauchy sequence in $\frac{X}{M}$. Now, $\exists n_1 \in \mathbb{N} \ni$ if $n,m \ge n_1 \Rightarrow \|(x_n+M)-(x_m+M)\|_q < \frac{1}{2}$. Also, $\exists n_2' \in \mathbb{N} \ni$ if $n,m \ge n_2' \Rightarrow \|(x_n+M)-(x_m+M)\|_q < \frac{1}{2^2}$. Choose $n_2 > \max\{n_1, n_2'\}$, we have $n_2 > n_1$ and $n_1, n_2 \ge n_1 \Rightarrow \|(x_{n_2}+M)-(x_{n_1}+M)\|_q < \frac{1}{2}$. Continuing this way we have a subsequence $\{x_{n_k}+M\}$ of $\{x_n+M\}$ such that $n_{k+1} > n_k$ and $\left\|(x_{n_{k+1}}+M)-(x_{n_k}+M)\right\|_q < \frac{1}{2^k}$. Now, choose $y_1 \in x_{n_1}+M$, then $y_1+M = x_{n_1}+M$ and since $\|(x_{n_2}+M)-(y_1+M)\|_q = \|(x_{n_2}+M)-(x_{n_1}+M)\|_q < \frac{1}{2}$, then there exist $y_2 \in x_{n_2}+M$ such that $\|y_2-y_1\| < \frac{1}{2}$. Proceeding in this way, we have a sequence $\{y_k\}$ in X such that $y_k+M = x_{n_k}+M$ and $\|y_{k+1}-y_k\| < \frac{1}{2^k} \quad \forall k \ge 1$. Let $k > r$, then

$$
\begin{aligned}
\|y_k-y_r\| &= \|(y_k-y_{k-1})+(y_{k-1}-y_{k-2})+\cdots+(y_{r+1}-y_r)\| \\
&\le \|(y_k-y_{k-1})\|+\|y_{k-1}-y_{k-2}\|+\cdots+\|y_{r+1}-y_r\| \\
&< \frac{1}{2^{k-1}}+\frac{1}{2^{k-2}}+\cdots+\frac{1}{2^r} \\
&< \frac{1}{2^{r-1}}
\end{aligned}
$$

Therefore $\{y_k\}$ is a Cauchy sequence in X. Since X is Banach space there is $y \in X$ such that $\lim_{k\to\infty} \|y_k-y\| = 0$. Now, $\left\|(x_{n_k}+M)-(y+M)\right\|_q = \|(y_k+M)-(y+M)\|_q = \|(y_k-y)+M\|_q \le \|y_k-y\| \to 0$ as $k \to \infty$. Hence $\lim_{k\to\infty}(x_{n_k}+M) = y+M \in \frac{X}{M}$. Now, the Cauchy sequence $\{x_n+M\}$ has a convergent subsequence in $\frac{X}{M}$. Hence $\lim_{n\to\infty}(x_n+M) = y+M \in \frac{X}{M}$. Thus $\frac{X}{M}$ is Banach space

EXERCISES FOR SECTION 1

In problems 1-5 prove that the given space is Banach space.

1. $l_\infty = \{x = \{x_n\}_{n=1}^{\infty} : \sup_{n\in\mathbb{N}} |x_n| < \infty, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ with the norm $\|\{x_n\}_{n=1}^{\infty}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
2. $c = \{x = \{x_n\}_{n=1}^{\infty} : \lim_{n\to\infty} x_n \in \mathbb{F}, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ with the norm $\|\{x_n\}_{n=1}^{\infty}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
3. $c_0 = \{x = \{x_n\}_{n=1}^{\infty} : \lim_{n\to\infty} x_n = 0, \quad x_n \in \mathbb{F}, \forall n \in \mathbb{N}\}$ with the norm $\|\{x_n\}_{n=1}^{\infty}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.
4. $L_p([a,b]) = \{f : [a,b] \to \mathbb{F} : \int_a^b |f|^p < \infty\}$ with the norm $\|f\|_p = \sqrt[p]{\int_a^b |f(x)|\, dx}$.
5. $C^1([0,1]) = \{f : [0,1] \to \mathbb{F} : f' \in C([0,1])\}$ with the norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|$.
6. Let X be a normed space and M be a closed subspace of X. Suppose that M and $\frac{X}{M}$ are Banach spaces. Prove that X is a Banach space.
7. Consider the Banach space $C([0,1])$ with the sup-norm and let $M = \{f \in C([0,1]) : f(0) = 0\}$ prove that M is closed subspace and $\frac{C([0,1])}{M} \cong \mathbb{F}$.